Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Models of dynamics of large quantal systems

Diploma thesis

2015 Mgr. Michal Širaň

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Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Branch of study: Physics
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Aim:

This thesis requires preliminary study of the mathematical formulation of (infinitely) large systems in the framework of non-relativistic quantum mechanics (infinite tensor products of Hilbert spaces, C*-algebras and their representations, automorphisms).

After this stage concrete models of quantal systems "with infinite number of degrees of freedom" will be investigated leading to either some type of irreversible behavior or furnishing an interesting time-evolution of their macroscopic observables.

In addition it is possible to discuss connections to some fundamental questions regarding quantum theory as well as physics in general.

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Modely dynamiky veľkých kvantových systémov

Cieľ: Práca si vyžaduje predbežné štúdium matematického opisu (aj nekonečne) veľkých systémov v rámci nerelativistickej
kvantovej mechaniky (nekonečné tenzorové súčiny Hilbertových priestorov, C*-algebry a ich reprezentácie, automorfizmy).
Potom sa budú vyšetrovať konkrétne modely kvantových systémov "s nekonečným počtom stupňov voľnosti" buď vedúce k nejakému typu nevratného správania, alebo vykazujúce zaujímavý časový vývoj ich makroskopických veličín.
Pritom sa asi vyskytne možnosť pojednať aj o niektorých fundamentálnych otázkach kvantovej teórie, resp. všeobecne fyziky.

Vedúci: doc. RNDr. Pavel Bóna, CSc.
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....
študent vedúci práce

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Finally and most importantly, I would like to thank my wife Mária *sine qua non*.

I declare that I have written this thesis myself using only the literature listed in the bibliography.

Bratislava, 18.5.2015 ....

Mgr. Michal Širaň

Abstract

Michal Širaň. Models of dynamics of large quantal systems. Diploma thesis. Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics, Department of Theoretical Physics and Didactics of Physics. Thesis Advisor: doc. Ing. Pavel Bóna, CSc. Bratislava, May 2015, 34 p.

This thesis deals with connections between "microscopic" and "macroscopic" systems in the framework of non-relativistic quantum mechanics. The emphasis is on the study of dynamics of these systems.

The focus of the first part is an overview of the results of the paper by P. Bóna and the present author. A finite spin chain model coupled to a Fermi field is constructed. The Hamiltonian of the composite system allows the spin chain to emit a scalar fermion which, given enough kinetic energy, has capability to escape into infinity. In the limit of infinite time and for sufficient length of the spin chain this model can be thought of as an "effective" model of a "measurement process" in the sense proposed by K. Hepp. It is also related to the decoherence approach to the resolution of the "measurement problem" of W. Zurek.

The second part focuses on a certain class of mean field theories arising as thermodynamic limits of copies of quantal systems given by a Lie group G of kinematic symmetries. The interaction between such copies is infinitely weak and infinitely long range. These models and their dynamics were studied in detail by P. Bóna. A generalization of some of the results for the case of a non-compact Lie group is presented. A central object of the considerations is a certain "mean field connection" which is a projector valued measure on the dual of the Lie algebra of G with values in the centre of the double dual of the C^*-algebra of the infinite quantal system. Particular emphasis is placed on the emergence of "macroscopic" observables and canonical state decompositions corresponding to them. Finally dynamics of the models are introduced for a large class of admissible Hamiltonians.

Keywords: quantum measurement, dynamics, mean field, thermodynamic limit

Abstrakt

Michal Širaň. Modely dynamiky veľkých kvantových systémov. Diplomová práca. Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, Katedra teoretickej fyziky a didaktiky fyziky. Vedúci diplomovej práce: doc. Ing. Pavel Bóna, CSc. Bratislava, máj 2015, 34 s.

Táto práca sa venuje vzťahu "mikrosokopických" a "makroskopických" systémov v rámci nerelativistickej kvantovej mechaniky. Dôraz je na štúdium dynamiky takýchto systémov.

Prvá časť sa zameriava na prehľad výsledkov článku P. Bónu a autora tejto diplomovej práce. Skonštruuje sa konečná spinová reťaz spojená s Fermiho poľom. Hamiltonián tohoto systému umožňuje reťazi vyžiariť skalárny fermión, ktorý pri dostatočne veľkej kinetickej energii môže uniknúť do nekonečna. V limite nekonečného času a pre dostatočne dlhú spinovú reťaz sa tento model dá pokladať za "efektívny" model "procesu merania" v zmysle navrhovanom K. Heppom. Model súvisí aj s prístupom dekoherencie k "problému merania" od W. Zureka.

Druhá časť sa zameriava na určitú triedu teórií so stredovým poľom, pochádzajúcich z termodynamických limít kópii kvantových systémov s danou Lieovou groupou G kinematických symetrií. Interakcia medzi týmito kópiami je nekonečne slabá a nekonečne ďalekodosahová. Tieto modely a ich dynamika boli podrobne študované P. Bónom. V práci ukážeme zovšeobecnenie niektorých výsledkov na prípad nekompaktnej Lieovej grupy. Hlavným objektom našich úvah je určitý pojem "konexie stredného poľa" ktorá je projektorovou mierou na duále Lieovej algebry Lieovej grupy G s hodnotami v centre druhého duálu C^*-algebry celého nekonečného kvantového systému. Špeciálne sa zameriame na objavenie sa "makroskopických" pozorovateľných a kanonické rozklady stavov zodpovedajúce týmto pozorovateľným. Nakoniec zavedieme dynamiku pre širokú triedu prípustných Hamiltoniánov.

Kľúčové slová: kvantové meranie, dynamika, stredné pole, termodynamická limita

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Introduction

This thesis consists of two Chapters which are independent yet they have a common theme. We focus on the exploration of connections between the (loosely defined) “microscopic” and “macroscopic” systems. A particular emphasis of our thesis is on the dynamics of these systems. Indispensable to our analysis is the rôle of large (and infinite) quantum systems for the “measurement process” and the emergence of “classical” quantities as well as their corresponding dynamics. The framework is that of rigorous mathematical physics. The algebraic approach to non-relativistic QM is especially stressed as well as its extensions to systems with infinite number of degrees of freedom. The tools are mostly the subjects of functional analysis and operator algebras techniques. Knowledge of standard material from these areas is assumed and can be found in Bratteli, Robinson [**12**], [**13**], Reed, Simon [**23**], [**24**], Megginson [**22**], Strocchi [**31**], von Neumann [**34**] and others.

Chapter 1 concerns a fundamental issue with various formulations of QM. In the literature it goes often under the name of the “measurement problem”, although there is little common agreement on the formulation and even less agreement on what constitues a resolution of the problem, e.g. see Schlosshauer, Kofler, Zeilinger [**28**]. In particular both the problem and resolutions are highly interpretation-dependent. Our approach will be that of what we call “orthodox QM”. By this term we mean that we consider QM to be an “irreducibly statistical” theory (excluding “hidden variables” interpretations) with the postulates of QM formulated by von Neumann [**34**] except that our approach is to work without the projection postulate and attempt to replace it with some version of the Schrödinger time-evolution using extensions of QM to systems with infinite number of degrees of freedom or by the use of decoherence approaches of Zurek [**35**]. Furthermore we consider QM to be a fundamental theory and therefore we would like to obtain a more satisfying postulates then those which use rather ambiguously the words “macroscopic” and “microscopic” in their formulation. Rather the emergence of classical phenomena, macroscopic apparati and observers should be at least in possibility a consequence of proper postulates of this “orthodox QM”.

The main focus of Chapter 1 is a short overview of the results of the paper Bóna, Širaň [**11**]. A model of finite spin chain with “domino dynamics” interacting with a Fermi field is constructed. The interaction term allows the spin chain to emit a scalar fermion (this is possible in non-relativistic QM and is technically easier to work with then the bosonic case). Given enough kinetic energy this fermion has capability to escape into infinity. As a “model of measurement” the interpretation of this “radiating spin chain” is as follows. The zeroth spin of the spin chain serves the rôle of the “measured microsystem”. The rest of the spin chain is a model of the “macroscopic measurement apparatus”. The rôle of the Fermi field is to account for the influence of the surrounding environment of the composite system consisting of the “measured microsystem” and “measuring macrosystem”. The crucial rate of convergence of the system for large time to a different macroscopic state corresponding to the emission and escape of the fermion is calculated. The probability of the fermion emission approaches unit almost exponentially as a function of time. In this sense our model describes a theoretical possibility of an effective description of the “measurement process” in the sense of Hepp [**17**].

Chapter 2 concerns quantal systems with infinite number of degrees of freedom described by a C^*-algebra $\mathfrak{A}^\Pi$ with an additional structure of an action σ of a connected locally compact Lie group G by *-automorphisms. It turns out that one can define what we call a “mean field connection” E which is a projector valued measure on $\mathfrak{g}^*$ with values in the centre of the double dual of $\mathfrak{A}^\Pi$. An important condition on E is that it is G-equivariant with respect to the (extended) coadjoint action of G on $\mathfrak{g}^*$. While these "mean field connections" may sound rather abstract they have a very nice interpretation in cases when $\mathfrak{A}^\Pi$ arises from a C^*-inductive limit (a colimit in the category of C^*-algebras) of copies of the same system given by a C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ equipped with an action σ which determines a projective representation U of G on $\mathcal{H}$ or unitary representation of a corresponding central extension of G on $\mathcal{H}$. The generators of these "kinematic symmetries" give rise in the colimit (by use of Cesaro means) to intensive "macroscopic" observables. They are the elements of the highly-nontrivial centre of $(\mathfrak{A}^\Pi)^{**}$. The mean field connection E arising from U and the colimit structure of $\mathfrak{A}^\Pi$ here is easily constructed using the SNAG theorem. This can be done in the case of bounded generators as in Bóna [**8**]. We will generalize this approach to unbounded generators. Some of the considerations appear in the unpublished work of Bóna [**10**].

For suitable classical Hamiltonians $Q \in C^\infty(\mathfrak{g}^*)$ the classical 1-parameter group of diffeomorphisms φ_t^Q (time-evolution) is well defined on the macroscopic phase space $\mathfrak{g}^*$ associated to $(\mathfrak{A}^\Pi, \sigma)$ leaving the (extended) coadjoint orbits invariant. A choice of a mean field connection E then determines the time-evolution τ_t^Q of both the observables of $\mathfrak{A}^\Pi$ and the classical observables in the centre of $(\mathfrak{A}^\Pi)^{**}$. It is crucial to consider these quantal and classical observables together in one single framework since the time-evolution τ_t^Q does not leave $\mathfrak{A}^\Pi$ invariant. This has been recognized in various physical models and is a rather general feature of quantal systems with infinite number of degrees of freedom. In order to define a rigorous time-evolution of such systems free of any apparent "paradoxes" it is often necessary to take into account the rôle of "macroscopic" observables. In the examples when $(\mathfrak{A}^\Pi, \sigma)$ arises as a C^*-inductive limit as above these "classical" observables emerge from the infinite Cesaro means of the generators and their time-evolution is governed by φ_t^Q. This can be thought of as the time-evolution of the resulting "mean field" of the infinite quantal system. The name "mean field" here derives from the fact that the time-evolution of any finite quantum subsystem can be described as evolving in the time-dependent Hamiltonian given by the interaction with the classical mean field. Let us stress here that this picture of dynamics is exact and is not an approximation. In both the bounded and unbounded generators case the time-evolution τ_t^Q can be shown to coincide with the local quantal time-evolutions of finite subsystems corresponding to the local Hamiltonians associated with Q and the unitary representation U.

The results of Chapter 2 for the locally compact group and unbounded generators have not been yet published. We conclude the Chapter with some lines of further research we would like to consider.

CHAPTER 1

A radiating spin chain as a model for quantum measurement

1. What the "measurement problem" is not

To discuss what the "measurement problem" is we first discuss what in our opinion the "measurement problem" is not. This question is highly intepretation sensitive. In order to formulate the problem one has to choose an interpretation of QM. In particular a crucial question is whether QM is an "irreducibly statistical" theory. We will assume this is so. This seems to be a widespread opinion as can be gathered from glancing in textbooks, articles and participating in conversations in the broad physics community. Let us make explicit that we do not believe that "hidden variables" intepretations of QM are ruled out by Bell's theorem. On the contrary there are known constructions of "hidden variables" interpretations and modifications of QM. We will tacitly ignore this discussion and stick to the "irreducibly statistical" intepretation. We want to make clear that a lot of confusion about the "measurement problem" seems to arise from adherents of the "irreducibly statistical" interpretation. Claims are made that decoherence or decoherence-like approach do not solve the "measurement problem". Since the goal of this Chapter is to present one such model in Section 4 we examine one particulary widespread claim. Its articulation is put forward in an oft-quoted work of Adler [**1**]. Let us present the claim.

Consider a measured microsystem $\mathcal{S}$ coupled to a "macroscopic" measurement apparatus $\mathcal{A}$ surrounded by the environment $\mathcal{E}$. We consider both $\mathcal{A}$ and $\mathcal{E}$ as quantal systems with large number of degrees of freedom. Let the measured microsystem $\mathcal{S}$ be a spin-1/2 system and let $\mathcal{S}$ be at $t = 0$ in the pure state

$$|\psi_0\rangle_{\mathcal{S}} := c_\downarrow|\downarrow\rangle_{\mathcal{S}} + c_\uparrow|\uparrow\rangle_{\mathcal{S}}. \tag{1}$$

It is a factor of the tensor product state

$$|\Psi_0\rangle_{\mathcal{S}+\mathcal{A}+\mathcal{E}} := |\psi_0\rangle_{\mathcal{S}}|\phi_0\rangle_{\mathcal{A}+\mathcal{E}}. \tag{2}$$

This state evolves unitarily by the Schrödinger equation to an entangled state in some physical experiments e.g. by usage of a Stern-Gerlach apparatus to

$$|\Psi_T\rangle_{\mathcal{S}+\mathcal{A}+\mathcal{E}} = c_\downarrow|\downarrow\rangle_{\mathcal{S}}|\phi_\downarrow\rangle_{\mathcal{A}+\mathcal{E}} + c_\uparrow|\uparrow\rangle_{\mathcal{S}}|\phi_\uparrow\rangle_{\mathcal{A}+\mathcal{E}} \tag{3}$$

where the states $|\phi_\downarrow\rangle_{\mathcal{A}+\mathcal{E}}$ resp. $|\phi_\uparrow\rangle_{\mathcal{A}+\mathcal{E}}$ correspond to the states of the apparatus (entangled to the environment) with "macroscopically" distinguishable pointer positions. We will not attempt to follow von Neumann [**34**] interpretation in terms of a regress to a conscious observer as this "resolution" does not seem to be plausible especially if multiple conscious observers are part of the described system. Besides if we believe that the world was evolving without any notion of a "collapse" until first sentient beings started observing, there is little reason not to proceed to some "universal wavefunction" approach. We will not follow this route.

The evolution from (2) to (3) as an admissible process in QM is we believe "non-negotiable" unless one wants to give up linearity of QM or change the common conceptions of "premeasurement" and "ideal measurement", again see von Neumann [**34**]. Decoherence is based on the following idea. If one has "effectively" orthogonal the two states

$$_{\mathcal{A}+\mathcal{E}}\langle\phi_\downarrow|\phi_\uparrow\rangle_{\mathcal{A}+\mathcal{E}} \approx 0 \tag{4}$$

and furthermore if it is impossible to measure entanglement of a huge number of particles then one effectively gets a statistical ensemble by tracing out the environment. This second if is as far as we know still an open and difficult problem. However the critique of Adler [**1**] does not take this into account, so we will for now ignore this issue. The "effective" equation (4) holds in the case of elastic instantaneous approximation and without back reaction of the measuring apparatus on the environment.

We need to stress one of the most distinguishing feature of QM at this point. The states of QM do not form a simplex. In particular a state which is not pure has infinitely many decompositions into extreme points of the convex w^*-compact set of states. This is already apparent in the spin-1/2 case when a general density matrix is a point on the Bloch ball which is a 3-dimensional ball of unit norm. Interestingly in Chapter 2 we will obtain a canonical decomposition using states of an emergent "macroscopic" system in Proposition 8. In the decoherence approach the environment induces superselection rules by the mechanism of "einselection", see Zurek [**35**]. Thus in the decoherence approach the resulting state admits an "environmentally selected" decomposition into "macroscopically distinguishable" pointer states.

Now the common critique of the decoherence approach to the "measurement problem" claim the failure of the project by the following consideration as in Adler [**1**]. Consider a "measurement" of a pure state e.g. a spin-1/2 in the initial state (1). We can describe this state equivalently by the following density matrix in the basis $\{|\downarrow\rangle_{\mathcal{S}}, |\uparrow\rangle_{\mathcal{S}}\}$

$$(\rho_0)_{\mathcal{S}} = \begin{pmatrix} |c_\downarrow|^2 & c_\downarrow^* c_\uparrow \\ c_\downarrow c_\uparrow^* & |c_\uparrow|^2 \end{pmatrix}$$

After the "measurement process" takes place we are told that the density matrix $(\rho_0)_{\mathcal{S}}$ "collapses" to the density matrix $(\rho_T)_{\mathcal{S}}$ which is with probability $|c_\downarrow|^2$ resp. $|c_\uparrow|^2$

$$(\rho_T)_{\mathcal{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ resp. } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

This is obviously too simplified - the above description should account for the inclusion of the measurement apparatus. Thus the matrix notation should correspond to states $|\downarrow\rangle_{\mathcal{S}}|\phi_\downarrow\rangle_{\mathcal{A}+\mathcal{E}}$ or $|\uparrow\rangle_{\mathcal{S}}|\phi_\uparrow\rangle_{\mathcal{A}+\mathcal{E}}$. Since we assume "effective" orthogonality (4) we actually have that these two results are "effectively" orthogonal which is impossible by the unitary time-evolution of the state (2). But by the above description of the decoherence process

$$(\rho_T)_{\mathcal{S}} = \mathrm{tr}_{\mathcal{E}}\left(|\Psi_T\rangle_{\mathcal{S}+\mathcal{A}+\mathcal{E}}\ {}_{\mathcal{S}+\mathcal{A}+\mathcal{E}}\langle\Psi_T|\right)$$

one actually gets

$$(\rho_T)_{\mathcal{S}} \approx \begin{pmatrix} |c_\downarrow|^2 & 0 \\ 0 & |c_\uparrow|^2 \end{pmatrix}. \tag{6}$$

effectively diagonalized with non-zero entries on the diagonal. This is a statistical mixture which does not correspond to a pure state. This is claimed to be contrary to the result of the measurement process.

But if we accept that QM is an "irreducibly statistical" theory it is impossible to obtain a description of a process which leads to either of the two results in (5). This would amount to an inclusion of some "hidden variables" mechanism. In an "irreducibly statistical" theory one cannot hope to go beyond an evolution to a state which is a classical statistical mixture as (6) supplemented with superselection rules.

The rest of this Chapter is organized as follows. In Section 2 we will contrast the unitary time-evolution given by the Schrödinger equation with the "measurement process" given first by a simple "projection postulate". We then describe some first attempts to model the "measurement process" as some kind of limit of the unitary time-evolution. It is well known that these attempts are bound to fail unless some conceptual change of the formulation of the "measurement problem" is done.

In Section 3 we only briefly recall the "QM domino" model of Bóna [**6**]. This is done since the main model of this Chapter will be a modification of the "QM domino". It is also important from the conceptual viewpoint. In order to model the "measurement process" it is possible to extend QM to systems with infinite number of degrees of freedom. As the "QM domino" model exhibits, it is then possible to model the "measurement process" by unitary time-evolution in the limit of infinite time.

Section 4 presents the setup of the finite radiating spin chain model. This is a summary of the work Bóna, Širaň [**11**] which is a detailed analysis of the finite radiating spin chain model first appearing in Bóna [**7**]. We will omit the technical proofs. The interested reader can find them in the corresponding cited paper.

We offer some concluding remarks in Section 5.

2. Unitary time-evolution (SchR-process) vs. measurement process (R-process)

Let us recall the sharp distinction between the unitary Schrödinger time-evolution (denoted here for brevity as SchR-process) and the non-unitary "measurement process" (denoted for brevity as R-process) featured in many formulations of QM. Interpretations of QM which do not account for this distinction have to explain the effective "appearance" of R-process. We recall proofs of certain "no-go" theorems which show that the R-process is not some limit version of the SchR-process. As is usual in physics "no-go" theorems often contain subtle assumptions which are often invisible in their formulation. Spotting them often amounts to deep reconsideration of the fundamentals of a given theory.[1] As we will see there are at least two such hidden assumptions in the proofs of the "no-go" theorems.

- They work only in the framework of QM of finite number of degrees of freedom, thus effectively eliminating any arguments made from e.g. thermodynamic limits, which are ubiquitous to physics.
- The unrealistic assertion of a closed isolated system of the measured microsystem and the macroscopic apparatus is made not accounting for the impact of the surrounding environment on the R-process.

We will present two models violating these assumptions in which R-process is modelled by a limit of SchR-process.

- The "QM domino" model of Bóna [**6**] is briefly sketched in Section 3. This model violates the first assumption since it is constructed from an infinite number of spin-1/2 systems. Its philosophy in terms of the "measurement problem" is close to that of Hepp [**17**].
- The second model is a finite radiating spin chain of Section 4. While the setup and dynamics are similar to that of the "QM domino" no extension of QM of system of finite number of degrees of freedom is needed. Instead it is the second assumption of closed isolated system which is violated. In particular the effect of the environment surrounding the composite system consisting of the measured microsystem and the measuring apparatus is taken into account by the possibility of the finite spin model to radiate a particle which can escape into infinity. The philosophy of this model is also close to that of Hepp [**17**] and in a way also to decoherence of Zurek [**35**]. There are subtle differences. In a sense the model exhibits a "true" decoherence in the sense that in the infinite time limit the dynamics is irreversible (the fermion escapes into infinity). This is not the case of decoherence approaches which yield almost periodic processes. In sharp contrast with the "QM domino" model of Bóna [**6**] we do not end up with a statistical ensemble of macrostates and consequently no superselection rules emerge. Instead if the chain is sufficiently long the resulting superposition can be "effectively" treated as a statistical ensemble. This model was first introduced by Bóna [**7**] and studied in detail in Bóna, Širaň [**11**].

Note that in both models the R-process is modelled as an infinite time limit of SchR-process. Given a sufficiently fast rate of convergence of the interference terms between the macroscopicaly distinguishable states to zero, one argues that such models are plausible.

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[1] A notorious example of this is a proof by Weinberg, Witten of the impossibility of descibing the graviton as a bound state of two spin-1 particles, which does not take into account the possibility of the holographic principle and appears to be refuted by the AdS/CFT correspondence.

We are now ready to give the relevant postulates of what we call "orthodox QM". In particular we will follow closely the account of von Neumann [**34**] and the Copenhagen interpretation. However a crucial missing part of these interpretations is the lack of clear criteria (without recurring to classical world of macroscopic conscious observers) of choosing between the SchR-process and R-process for the evolution of a given quantum system. Connected to this issue is an often tacit assumption in postulates of these formulation of QM of a boundary between "microscopic" and "macroscopic" systems. In particular what we loosely call as "orthodox QM" has the ambition to eliminate this boundary by treating both "microscopic" and "macroscopic" systems using the same set of QM laws. We call it "orthodox" since it retains the irreducibly statistical character of QM. In a sense Chapter 1 deals with an attempt to remove the R-process from the postulates of formulations of QM. Chapter 2 will treat the emergence of some classical "macroscopic" observables along with consistent dynamics from considerations of the underlying quantal system.

Consider a QM system with the Hilbert space of states $\mathcal{H}$ and the observables given by the set of selfadjoint (possibly unbounded) operators denoted $\mathcal{L}_{sa}(\mathcal{H})$. In addition to the pure states (denoted by $|\psi\rangle, |\psi'\rangle \in \mathcal{H}$) we also consider the more general states given by density matrices (denoted by $\rho, \rho' \in \mathcal{L}(\mathcal{H})$. Let $\mathcal{B}(\mathbb{R})$ be the $\sigma-$algebra of Borel subsets of $\mathbb{R}$. Given a selfadjoint operator $x \in \mathcal{L}_{sa}(\mathcal{H})$ we denote

$$E_x(\cdot) : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_{sa}(\mathcal{H})$$

to be the spectral measure of x. As usual $\sigma(x)$ denotes the spectrum of x. Let us recall that there are two types of time-evolutions in the formulations of QM which we would like to modify - especially the von Neumann [**34**] and the Copenhagen interpretation.

Definition 1. *(SchR-process and R-process)*

- *The unitary time-evolution SchR-process is given by the Schrödinger equation - for a general non-conservative system given by a time-dependent Hamiltonian $t \mapsto H(t) \in \mathcal{L}_{sa}(\mathcal{H})$ the equation*

$$i\frac{d}{dt}\rho_t = H(t)\rho_t - \rho_t H(t) \quad \textit{resp.} \quad i\frac{d}{dt}|\psi_t\rangle = H(t)|\psi_t\rangle. \tag{7}$$

- *The R-process describes the measurement process of a given observable x with a spectral decomposition*

$$x = \int_{\sigma(x)} \lambda E_x(d\lambda) \in \mathcal{L}_{sa}(\mathcal{H}),$$

by which the probability of the outcome of the measurement of x in the state ρ resp. $|\psi\rangle$ lying in the (Borel) set $U \subset \mathbb{R}$ is

$$Pr(U, x; \rho) = Tr(E_x(U)\rho), \quad \textit{resp.} \quad Pr(U, x; |\psi\rangle) = \|E_x(U)|\psi\rangle\|^2,$$

and if the measurement result lies in the given set U the state of the quantal system after the measurement is

$$\rho' = \frac{E_x(U)\rho E_x(U)}{Tr(E_x(U)\rho)}, \quad \textit{resp.} \quad \psi' = \frac{E_x(U)|\psi\rangle}{\|E_x(U)|\psi\rangle\|}.$$

As was extensively argued above and since the present author's confession is "orthodox QM" as an "irreducibly statistical" theory without "hidden variables" the correct notion of R-process corresponding to measurement of observable x is then a process which takes the form

$$\rho \mapsto \rho' = \sum_k E_x(U_k)\rho E_x(U_k) \tag{8}$$

where $\{U_k\}$ form a (countable) non-overlapping cover of $\sigma(x)$. Here we account for certain imprecision in the measurement so that points of spectra in U_k are for practical purposes indistinguishable. This is done to formulate the R-process for measurement of x with continuous spectrum. In the case of a point spectrum (8) reduces to the sum $\sum_k P_k\rho P_k$ of orthogonal projectors corresponding to the point spectrum.

In usual descriptions of axioms of QM this process is defined as instanteneous.

It is natural to ask (in the spirit of Occam's razor, or more precisely Aquinas' razor) whether these two processes are distinct. Obviously one needs to modify the R-process (8) instanteneousity in order for it to be some version of SchR-process (7). So we will allow the R-process (8) to be a process which takes some reasonably small time. In applications it appears that $t \leq 10^{-5}s$ is a reasonable bound, see Leggett [**19**].

There are many types of "no-go" theorems claiming that R-process is distinct from limits of SchR-process. We will present two of them which are of importance. The first one is easy to prove and claims that R-process is certainly not a SchR-process, as can be seen from unitarity considerations (and thus irreversibility considerations).

We will utilize the Hilbert-Schmidt inner product defined by $(\rho, \rho')_{HS} = \mathrm{Tr}(\rho^* \rho')$ on the class of Hilbert-Schmidt operators. The following considerations are standard and can be found in a variety of textbooks. For a more detailed discussion see Bóna [**5**]. Let us sketch the first argument.

PROPOSITION 1. *Let $\rho \mapsto \rho'$ be a R-process. Then if it is not an identical process ($\rho' = \rho$) it is not unitary.*

PROOF. Consider a R-process $\rho \mapsto \rho'$ and suppose to the contrary that it is unitary and non-identical so that we can write

$$\rho' := U \rho U^* \neq \rho.$$

Unitarity implies that the Hilbert-Schmidt norms of ρ, ρ' are equal since $(\rho, \rho)_{HS} = (\rho', \rho')_{HS}$. Let us from now on write $\rho_{\mathrm{pt}}, \rho'_{\mathrm{pt}}$ for the point spectrum projections to the point spectrum of the measured observable

$$\rho_{\mathrm{pt}} = P_{\mathrm{pt}} \rho P_{\mathrm{pt}}, \quad \rho'_{\mathrm{pt}} = P_{\mathrm{pt}} \rho' P_{\mathrm{pt}}$$

(a more general proof is possible without restriction to point spectra). From the description of the R-process as above we have that projecting to point spectrum of the measure

$$\rho'_{\mathrm{pt}} := \sum_k P_k \rho_{\mathrm{pt}} P_k,$$

so we can write the (square) of the Hilbert-Schmidt norm as

$$(\rho'_{\mathrm{pt}}, \rho'_{\mathrm{pt}})_{HS} = \sum_{k,l} \mathrm{Tr}(P_k \rho_{\mathrm{pt}} P_k P_l \rho_{\mathrm{pt}} P_l) = \sum_k \mathrm{Tr}(P_k \rho_{\mathrm{pt}} P_k \rho_{\mathrm{pt}}),$$

which implies

$$(\rho'_{\mathrm{pt}}, \rho'_{\mathrm{pt}})_{HS} = (\rho'_{\mathrm{pt}}, \rho_{\mathrm{pt}})_{HS}.$$

Using Schwarz inequality

$$|(\rho_{\mathrm{pt}}, \rho'_{\mathrm{pt}})|^2_{HS} \leq (\rho_{\mathrm{pt}}, \rho_{\mathrm{pt}})_{HS} (\rho'_{\mathrm{pt}}, \rho'_{\mathrm{pt}})_{HS}$$

reveals that

$$(\rho'_{\mathrm{pt}}, \rho'_{\mathrm{pt}})_{HS} \leq (\rho_{\mathrm{pt}}, \rho_{\mathrm{pt}})_{HS}.$$

In the last equation equality is possible only for $\rho' = C\rho$ and we get $C = 1$ for the normalization, hence a contradiction. □

REMARK. Let us note that we obtained that for the R-process $\rho \mapsto \rho'$ the Hilbert-Schmidt norm decreases

$$\|\rho'\|_{HS} < \|\rho\|_{HS},$$

so the R-process is irreversible. □

There are several threads of thought how one could still hope to obtain R-process as a sort of "disguised" SchR-process. The one of interest to us is to attempt to construct a R-process as a weak limit of SchR-process. In more detail we consider $\rho \mapsto \rho'$ given by an infinite-time limit (in some suitable operator topology) of the SchR time-evolution given by a weakly-continuous 1-parameter group (by Stone's theorem) $U_t = \exp(-itH)$, where H is the Hamiltonian of the combined QM system (usually constructed from the measured microsystem $\mathcal{S}$ and the macroscopic measurement apparatus $\mathcal{A}$). We can therefore write

$$\rho' := (\text{some topology}) - \lim_{t \to \infty} U_t \rho U_t^* := (\text{some topology}) - \lim_{t \to \infty} \rho_t. \tag{9}$$

PROPOSITION 2. *The R-process cannot be modeled by the limit* (9) *for norm, strong or weak topology.*

PROOF. It is clear that the norm-topology will not suffice as it contradicts the irreversibility given by $\|\rho'\|_{HS} < \|\rho\|_{HS}$. The proof for the weak topology (from which the case of strong topology follows) can be found in Bóna [**5**]. □

3. Infinite QM domino SchR-process as a model of R-process

The above proofs while rigorous make a number of subtle assumptions which can be (and should be) questioned. As noted they are framed in the mathematical formulation of QM which handles systems with finite number of degrees of freedom. This seems like a reasonable framework (excluding now QFT as a more fundamental underlying theory which has infinite number of degrees of freedom by definition). After all our measured microsystems and macroscopic apparati should follow the rules of QM and even though the measurement apparati have huge number of degrees of freedom they are still finite. One can thus question the "ontological reality" of a system with infinite number of degrees of freedom to model R-process. We will leave this philosophical issue behind. For a working physicist (to paraphrase a term coined by MacLane [**21**]) infinite limits are his "panis cotidianum". Let us mention three examples which are probably indispensable to most theoretical physicists.

- The thermodynamic limit in statistical physics, condensed matter theory, and many other areas is fundamental to treat phenomena like phase transitions. In Chapter 2 the thermodynamic limits of a some QM systems will be our main focus.
- The infinite time-evolution limit in both the future and past direction in the definition of the S-matrix. Indeed without rotating out the non-interaction part of dynamics one is even unable to define the Fock state space of interacting QFT. It should be noted however, that S-matrix formalism is not applicable for interacting conformal field theories. Yet some CFTs are related with critical phenomena so we face infinity again. Still the particle interpretation of many QFTs such as QED, QCD, etc. depends crucially on the S-matrix formalism and in addition particles make sense only when considered on unbounded regions of spacetimes. There does not exist any number operator on any bounded open subset of Minkowski spacetime by the Reeh, Schlieder result [**26**].
- The usage of $\mathbb{R}$ in physics is indispensable yet it is a construction involving limits.

An extension of QM is needed to handle infinite systems. There are two possible paths of extending QM.

- From the level of "phase space", using infinite completed tensor products as described by von Neumann. One can easily construct infinite quantal systems by combining various finite quantal systems using the tensor product and taking the von Neumann complete tensor product space [**33**]. In Chapter 2 this construction will appear in Example 1.
- From the level of observables. This is conceptually simpler as the formulation of QM remains unchanged - as in the finite case the algebra of observables is a C^*-algebra. In QM of finite number of degrees of freedom these are built up from matrix algebras or (by Stone-von Neumann theorem) $\mathcal{L}\big(L^2(\mathbb{R}^3, d^3x)\big)$. The passage to quantal systems of infinite number of degrees of freedom can be then obtained by considering more general C^*-algebras which admit unitarily inequivalent irreducible representations. The above example involving tensor products is a special case of a more general scheme where one constructs the quantal system as a limit of QM systems of finite number of degrees of freedom. This can be described efficiently using the language of categories as an inductive limit in the category of C^*-algebras and *-morphisms. The resulting C^*-algebra of the limit quantal system is then a colimit or C^*-inductive limit
$$\mathfrak{A}^{\Pi} := \lim_{\overrightarrow{I}} \mathfrak{A}^{I}$$
where $\{\mathfrak{A}^I\}$ is the the net of local C^*-algebras of finite subsystems (the reader not familiar with limits and colimits in categories can consult MacLane [**21**]).

There are subtleties to these extensions. For example the limit quantal systems can fail to have a well defined Hamiltonian H but one can still define non-Hamiltonian time-evolution as a 1-paramteric group of *-automorphisms of $\mathfrak{A}^\Pi$. As noted $\mathfrak{A}^\Pi$ will have unitarily inequivalent irreducible GNS-representations which further complicates working with concrete representations. The origin and meaning of these "anomalies" will be discussed in Chapter 2.

The "QM domino" model was introduced by Bóna [**6**] and it shows that the impossibility of modelling the R-process as an SchR-process no longer holds when we consider infinite QM. The model consists of infinitely many spin-1/2 systems which can be imagined to be ordered in a chain. The idea is that the first spin is to play the rôle of the "measured microsystem" which we label $\mathcal{S}$. The rest of the chain (labeled by $n = 2, \dots$) serves the rôle of the "measuring apparatus" labelled $\mathcal{A}$. The Hamiltonian of this apparatus is given by the weak limit

$$H_\infty := w - \lim_{N\to\infty} \sum_{n=2}^{N} a_n^* a_n (a_{n+1}^* + a_{n+1}) a_{n+2} a_{n+2}^*,$$

where a_n^*, a_n are the creation and annihilation operators of the $n-$th spin, satisfying the canonical (anti)commutation relations for $n \neq m$

$$[a_n, a_m^*] = 0, \quad a_n a_n = 0, \quad \{a_n, a_n^*\} = 1. \tag{10}$$

The Hilbert space of the "measuring apparatus" is given as $\mathcal{H}_\mathcal{A} := \otimes_{n=2}^\infty \mathbb{C}^2$ and the interaction with the "measured microsystem" - the first spin of the chain - is given by the interaction part of the Hamiltonian

$$H_I := a_1^* a_1 (a_2^* + a_2) a_3 a_3^*.$$

It is easy to see why this model is called "QM domino". The dynamics of the spin chain is as follows - if the $n-$th spin is pointed up $|\uparrow\rangle$ and the $(n+1)$-th and $(n+2)$-th spins are pointed down $|\downarrow\rangle$ the $(n+1)$-th spin flips up

$$|\dots \uparrow\downarrow\downarrow \dots\rangle \mapsto |\dots \uparrow\uparrow\downarrow \dots\rangle.$$

The h.c. terms of the Hamiltonian $H = H_I + H_\infty$ make possible the reverse process. This dynamics is to be interpreted in the usual QM fashion involving superpositions of the up and down positions.

Coupling the state of the first spin $c_\downarrow|\downarrow\rangle + c_\uparrow|\uparrow\rangle$ to the rest of the spin chain in the state $|\downarrow \dots\rangle$ we obtain a state of the system $\mathcal{S} + \mathcal{A}$

$$|\phi\rangle := c_\downarrow|\downarrow\rangle|\downarrow \dots\rangle + c_\uparrow|\uparrow\rangle|\downarrow \dots\rangle$$

and the time-evolution of this state is $\phi_t := \exp(-itH)\phi$. For brevity of notation let $|\beta_0\rangle$ be the state of all spins pointing down and the states with the first m spins up will be denoted

$$|\beta_m\rangle := |\uparrow\rangle|\underbrace{\uparrow \dots \uparrow}_{m-1}\downarrow \dots\rangle.$$

Let $\mathfrak{A}$ be the corresponding C^*-algebra of the system $\mathcal{S} + \mathcal{A}$ and let $\mathcal{S}(\mathfrak{A})$ be the set of positive normalized linear functional on $\mathfrak{A}$. Denote $\omega_{|\phi_t\rangle} \in \mathcal{S}(\mathfrak{A})$ the state corresponding to $|\phi_t\rangle$. Using the GNS representation induced by $\omega_{|\phi_t\rangle}$ the expectation value of any local observable x in this state is

$$\begin{aligned}\omega_{\phi_t}(x) := \langle\phi_t|x|\phi_t\rangle = |c_\downarrow|^2 \langle\beta_0|\exp(itH)\ x\ \exp(-itH)|\beta_0\rangle \\ + |c_\uparrow|^2 \langle\beta_1|\exp(itH)\ x\ \exp(-itH)|\beta_1\rangle + 2\mathrm{Re}(\bar{c}_\uparrow c_\downarrow \langle\beta_0|\ x\ \exp(-itH)|\beta_1\rangle).\end{aligned}$$

Furthermore it can be shown (see Bóna [**6**] for details) that the interference term goes to zero as

$$|(\bar{c}_\uparrow c_\downarrow \langle\beta_0|\ x\ \exp(-itH)|\beta_1\rangle|^2 = O(t^{-3}).$$

In the limit of infinite time the interference thus disappears completely.

The quasilocal C^*-algebra of observables has infinitely many unitarily inequivalent representations. We can construct two unitarily inequivalent GNS representations as follows. Consider the states

$$\lim_{t\to\infty} \exp(-itH)|\beta_0\rangle = |\downarrow\rangle|\downarrow\downarrow\downarrow \dots\rangle, \quad \lim_{t\to\infty} \exp(-itH)|\beta_1\rangle = |\uparrow\rangle|\uparrow\uparrow\uparrow \dots\rangle,$$

which define two states $\omega_\downarrow, \omega_\uparrow \in \mathcal{S}(\mathfrak{A})$

$$\omega_\downarrow(x) := \lim_{t\to\infty} \langle\beta_0|\exp(itH)\ x\ \exp(-itH)|\beta_0\rangle, \quad \omega_\uparrow(x) := \lim_{t\to\infty} \langle\beta_1|\exp(itH)\ x\ \exp(-itH)|\beta_1\rangle.$$

In the weak topology limit of Cesaro mean "the average of the number of spins pointing up" defined as

$$M := \mathrm{w} - \lim_{t\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n^* a_n,$$

we get the expectation values

$$\omega_\downarrow(M) = 0, \quad \omega_\uparrow(M) = 1,$$

which can be interpreted as the macroscopic pointer on the measurement apparatus $\mathcal{A}$.

4. Finite radiating spin chain SchR-process as a model of R-process

Now we will describe another spin-1/2 chain model which appeared first in Bóna [**7**] and was studied in detail in Bóna, Širaň [**11**]. It is a composite model consisting of a spin-1/2 chain as in the "QM domino" of previous Section but this time of finite lenght $N+1$. Here we index the spin chain by $i = 0, \dots, N$ so that the chain begins with the zeroth spin. The dynamics governing the spin chain are the same one as in the previous Section except that additional interaction term is present which couples the end of the spin chain to a scalar Fermi field. In particular we interpret the last N-th spin-1/2 as a two-level unstable particle.

Let us consider the time-evolution of the initial state of all the spins pointing down and the Fermi field in the vacuum state. If the first spin flips, either by hand, or by some external influence such as a scattering with another quantal system, the "domino" dynamics of the previous model follows which subsequently flips all the following spins of the chain. Flipping of the final spin results via the interaction in an emission of a fermion by the chain. This interaction of the spin chain and the Fermi field is short-range so that the fermion with sufficiently large kinetic energy can escape irreversibly into infinity. The spin chain remains in the state with all $N+1$ spins pointing up in the limit $t \to \infty$. As usual this description is to be interpreted in the framework of QM in terms of superpositions.

This dynamics of this model can be interpreted as an "effective" description of the R-process. The "measured object" is the zeroth spin of the chain, the "measuring apparatus" is the rest of the finite spin chain and the Fermi field plays the rôle of the "environment". It should be noted that unlike the infinite spin chain model this radiating spin chain exhibits interference of "approximately" macroscopically different states. As we make the chain bigger by changing the size N of the apparatus this interference becomes apparently less probable, see Leggett [**19**] and Hepp [**17**]. It should be noted as carefully argumented by Hepp [**17**] that some process of this "decoherence type" appears to be the only possibility of the description of R-process in the framework of standard QM for systems of finite number of degrees of freedom.

Let us start first with a mathematical description of the model. The Hilbert space of states is

$$\mathcal{H} := (\mathbb{C}^2)^{N+1} \otimes \mathcal{F},$$

where $\mathcal{F}$ is the Fermi Fock space with the vacuum vector $|0\rangle_\mathcal{F}$ - the representation space of the canonical anticommutation relations C^*-algebra $\mathfrak{A}_\mathcal{F}$. For the spin space $(\mathbb{C}^2)^{N+1}$ we define $|0\rangle_\mathcal{S}$ to be the vacuum vector. It corresponds to the state of all spins pointing down.

The spin-1/2 creation and annihilation operators $a_i^*,\ a_i,\ i = 0, 1, \dots, N$ satisfying the anti-commutation relations (10) for $i, j = 0, 1, \dots, N$ acting on the space $(\mathbb{C}^2)^{N+1}$ generate the finite dimensional algebra of spin observables $\mathfrak{A}_\mathcal{S}$. The vacuum state $\omega_0 \in \mathcal{S}(\mathfrak{A}_\mathcal{S}) \subset (\mathfrak{A}_\mathcal{S})^*$ is the state satisfying $\omega_0(a_i^* a_i) = 0,\ i = 0, 1, \dots, N$. It defines the cyclic vacuum vector $|0\rangle_\mathcal{S}$, such that $\omega_0(a) = {}_\mathcal{S}\langle 0|a|0\rangle_\mathcal{S},\ a \in \mathfrak{A}_\mathcal{S}$. Since the spin chain emits a fermion all the creation and annihilation operators in the Hamiltonian will be bounded, see Bratteli, Robinson [**13**]. The self-adjoint Hamiltonian H on $\mathcal{H}$ is defined as a sum $H := H_0 + V$. The first operator is

$$H_0 := \left(\sum_{n=0}^{N-2} a_n^* a_n (a_{n+1}^* + a_{n+1}) a_{n+2} a_{n+2}^* - \varepsilon_0 a_N^* a_N \right) \otimes \mathrm{id}_\mathcal{F} + \mathrm{id}_\mathcal{S} \otimes d\Gamma(h), \tag{11}$$

where $\varepsilon_0 > 0$, and h is a self-adjoint operator on $L^2(\mathbb{R}^3, d^3x)$ that operates under the Fourier transform as

$$\widehat{h\phi}(\vec{p}) := \varepsilon(\vec{p})\widehat{\phi}(\vec{p}), \tag{12}$$

with $\varepsilon : \mathbb{R}^3 \to \mathbb{R}_+$ is to be specified later and where $\phi \in L^2(\mathbb{R}^3, d^3x)$ is arbitrary and the Fourier transform is defined with the following normalization

$$\widehat{\phi}(\vec{p}) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\vec{p}.\vec{x}} \phi(\vec{x}) d^3x,$$

Finally $d\Gamma(h)$ in (11) is the second quantization of h, see Bratteli, Robinson [**13**]. The interaction part of the Hamiltonian is

$$V := v^2 \left(a^*_{N-1} a_{N-1} a^*_N \otimes b^*(\sigma) + a^*_{N-1} a_{N-1} a_N \otimes b(\sigma) \right) \tag{13}$$

here $v \in \mathbb{R}$ and $\sigma \in L^2(\mathbb{R}^3, d^3x)$ are to be specified later, the (anti-)linear mappings

$$b, b^* : L^2(\mathbb{R}^3, d^3x) \to \mathcal{L}(\mathcal{F})$$

are the annihilation, resp. creation operators of the scalar fermion satisfying the canonical anti-commutation relations:

$$b(\varphi)^2 = 0, \; b(\varphi)b^*(\psi) + b^*(\psi)b(\varphi) = (\varphi, \psi)\mathrm{id}_{\mathcal{F}},$$

To ease the notation let us define the closed subspace $\mathcal{H}_1 \subset \mathcal{H}$

$$\mathcal{H}_1 := \overline{\mathrm{span}\{|0\rangle_{\mathcal{S}}|0\rangle_{\mathcal{F}}, |\beta_n\rangle, |\beta_N(\phi)\rangle; n = 0, \ldots, N-1, \phi \in L^2(\mathbb{R}^3, d^3x)\}},$$

where

$$|\beta_n\rangle := a^*_0 \ldots a^*_n |0\rangle_{\mathcal{S}}|0\rangle_{\mathcal{F}} = |\underbrace{\uparrow \ldots \uparrow}_{n+1} \downarrow \ldots \downarrow\rangle_{\mathcal{S}}|0\rangle_{\mathcal{F}},$$

$$|\beta_N(\phi)\rangle := a^*_0 \ldots a^*_N |0\rangle_{\mathcal{S}} b^*(\phi)|0\rangle_{\mathcal{F}} = |\uparrow \ldots \uparrow\rangle_{\mathcal{S}}|\phi\rangle_{\mathcal{F}}.$$

As is clear from the Hamiltonian the dynamics of the model is chosen such that the last N-th spin of the chain can emit or absorb a fermion in the state $|\sigma\rangle_{\mathcal{F}}$ where $\sigma \in L^2(\mathbb{R}^3, d^3x)$. The emission resp. absorbtion is coupled to the "flipping up" resp. "flipping down" of the N-th spin. This process is only possible if the $(N-1)$-st spin is "pointing up". This means that if the initial state is described by a state in $\mathcal{H}_1$, then the probability of the emission of a fermion by the chain (being in $\mathcal{H}_1$) equals the probability of the N-th spin to be "pointing up". We will show that the probability of the emission of a fermion by the chain starting in the inital state of the system corresponding to the vector $|\beta_n\rangle$, $n = 0, \ldots, N-1$ evolves with time $t \to \infty$ rapidly to 1. The speed of this convergence is "almost exponential". More precisely, we will show that for specific "conveniently chosen" parameters of the model the probability of the state with "all the spins pointing up" is

$$\langle\beta_n|\exp(itH)a^*_N a_N \exp(-itH)|\beta_n\rangle = 1 - o(t^{-m}),$$

for all $m \in \mathbb{N}$ and $t \to \infty$.

In order to do this we undergo an analysis of the Fourier transforms of matrix elements of the unitary group $\exp(-itH)$. A priori we do not know anything about integrability and the existence of classical Fourier transform of matrix elements $\langle\psi|\exp(-itH)|\varphi\rangle$ as functions of t. We will consider them as tempered distributions defined by locally integrable functions and work with the Fourier transform of these distributions instead. Effective theorems are valid for Fourier transforms of distributions defined on half-line or, in more dimension, on a cone. The Fourier transforms of such distributions in real domain are boundary values on $\mathbb{R}$ of complex analytic functions on an open half-plane, see Reed, Simon [**24**]. After a choice of model parameters we will obtain Fourier transforms of functions of the desired behavior.

If we choose the unspecified parameters of H so that for $a > 0$

$$\varepsilon(\vec{p}) := a|\vec{p}|^2$$

the following equation will be satisfied

$$\lim_{t\to\infty} \langle\beta_0|\exp(itH)|\beta_n\rangle = 0$$

for $n = 0, \ldots, N-1$ and furthermore this convergence is as fast as possible. We choose $\sigma \in L^2(\mathbb{R}^3, d^3x)$ such that for some $b > 0$ we have $\widehat{\sigma} \in \mathcal{S}(\mathbb{R}^3)$ and

$$\widehat{\sigma}(\vec{p}) \begin{cases} = 0 & |\vec{p}| < b, \\ > 0 & |\vec{p}| > b. \end{cases}$$

As an example we can choose σ by the following

$$\mathcal{F}[\sigma](\vec{p}) := \begin{cases} 0 & |\vec{p}| < b, \\ C_1\, \omega_\delta(\vec{q}) * (e^{-\alpha|\vec{q}|^2}\theta(|\vec{q}| - b - \delta))(\vec{p}) & |\vec{p}| > b, \end{cases}$$

$$\omega_\delta(\vec{q}) := \begin{cases} \frac{1}{\delta} C_2\, e^{\frac{1}{|\vec{q}|^2/\delta^2 - 1}} & |\vec{q}| < \delta, \\ 0 & |\vec{q}| > \delta, \end{cases}$$

with $\delta, C_1, C_2 > 0$ being constants and the function ω_δ is the standard mollifier.

THEOREM 1. *Let the model be defined by (11), (12), (13) with any $\varepsilon_0 > ab^2 + 2$ (with possibly one exception), resp. with any*

$$\varepsilon_0 > 2 + ab^2 + 2v^4 \int_{ab^2}^{\infty} \frac{2\pi\sqrt{\varepsilon}|\widehat{\sigma}(\sqrt{\varepsilon/a})|^2 d\varepsilon}{a^{3/2}(\varepsilon - ab^2)}$$

and σ, ε as above. If the Fermi field is initially in the vacuum state and the first $n \geq 0$ spins are initially turned up, the time evolution of the probability of all the $N+1$ spins flipping up (realizing the final state of the spin chain) approaches unity "almost exponentially fast"

$$\langle \beta_n | exp(itH) a_N^* a_N \, exp(-itH) | \beta_n \rangle = 1 - o(t^{-m})$$

for any $m \in \mathbb{N}$.

PROOF. The proof is long and involved and can be found in Bóna, Širaň [**11**]. □

5. Concluding remarks

The dynamics of the radiating spin chain model is an "effectively" irreversible process. In contrast with the "QM domino" the time evolution of its "macroscopic pointer state" is much faster (in principle it is the fastest possible convergence). Another virtue is that the spin chain of this model is of finite length as is expected of a correct model of a "macroscopic apparatus". The almost periodical time-evolution of the finite spin chain is eliminated by the capability to radiate a fermion. In this point this model contrasts the usual decoherence models.

Since exact time-reversing is "practically impossible" after the emission of a fermion, this model can be thought of as a model of R-process. There are two distinguished stationary states of the system - the first one is the state of all spins down with the Fermi field in vacuum state. The second state is all spins up and thanks to the short range interaction and other choices of the parameters of the model the spin chain radiates a fermion which escapes to infinity so that the Fermi field is again in vacuum. For sufficiently large chain these two states are "effectively macroscopically distinguishable" in the sense of Hepp [**17**]. It is still theoretically possible to observe interference of these two "macroscopic" states using an observable with a nonzero matrix element between these two states corresponding to N-spin correlation. A key question is practical possibility for large N of an apparatus capable of measuring this N-spin correlation. This remains an open question for further study. While it seems plausible one needs to be careful of experimental results with quantum interference between macroscopically distinct states, see Leggett [**19**].

CHAPTER 2

Mean field theories with locally compact symmetry group

1. Introduction and algebraic approach to QM

QM systems with finite number of degrees of freedom are usually described in terms of concrete representations - using concrete Hilbert spaces as state spaces, (un)bounded self-adjoint operators on these Hilbert spaces as observables and density matrices as arbitrary states. The dynamics is usually given by a self-adjoint Hamiltonian or equivalently (Stone's theorem [**23**]) by a strongly-continuous 1-parameter group of unitary operators. A concrete representation often obscurs the importance of the algebraic structure of observables, the C^*-algebras (or von Neumann algebras) and the "algebraic dynamics" as 1-parameter groups (or semigroups in case of irreversible processes) of *-automorphisms of these C^*-algebras. For systems with finite number of degrees of freedom one can justify the concrete approach since the Stone-von Neumann theorem [**31**] says that for the C^*-algebras appearing in such systems there is only one (regular) irreducible representation up to unitary equivalence. This is either some matrix representation or the Schrödinger representation.

This framework does not handle "thermodynamic limits". For our purposes a "thermodynamic limit" will be loosely defined as a "limit" of a net of systems with finite number of degrees of freedom (see the next Section for a precise notion of this "limit"). It is well known that the time-evolution is often impossible to define using the "non-algebraic" approach for systems with infinite number of degrees of freedom as some "limit" of the dynamics of the corresponding finite subsystems. One virtue of the algebraic approach is that this problem does not arise. In fact the algebraic approach helps to explain the perplexity of the "non-algebraic" approach and the non-existence of time-evolution in concrete representations as a source of deeper physical phenomena. One example is the existence of phase transitions. Another is the appearance of "macroscopic" observables which play crucial rôle in interpretation of quantum theory.

The Stone-von Neumann theorem not valid in general for systems with infinite number of degrees of freedom. For a general system there exist unitarily inequivalent irreducible representations of the C^*-algebra of observables (of the algebra of canonical commutation relations or others). Such irreducible representations describe also different thermodynamical "phases" in equilibrium statistical physics. Switching between different unitarily inequivalent representations can amount to a phase transition. We see that one should work on the abstract level of the C^*-algebra instead of a concrete irreducible representation to define the dynamics. Otherwise we will not have a correct description of a phase transition. The other possibility is to work in a specific reducible representation suited for the case.

Let us be a little more specific. Consider a C^*-algebra $\mathfrak{A}$ of some system with infinite number of degrees of freedom. Two examples to keep in mind are the two-dimensional Ising spin model and an ideal Bose gas made up of infinite number of non-interacting spin-0 particles. To work concretely one has to choose a state of the system. A state in the C^*-algebraic framework is a positive, normalized, linear functional on $\mathfrak{A}$ and we denote the set of states $\mathcal{S}(\mathfrak{A})$. This is a convex w^*-compact subset of $\mathfrak{A}^*$. For any state $\omega \in \mathcal{S}(\mathfrak{A})$ the GNS theorem (e.g. Strocchi [**31**]) then constructs a C^*-algebra representation of $\mathfrak{A}$ that is a triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ where $\mathcal{H}_\omega$ is the Hilbert space of our representation,

$$\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{H}_\omega)$$

is a morphism of C^*-algebras and $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ is a cyclic vector for π_ω so that

$$\overline{\{\pi_\omega(\mathfrak{A})|\Omega_\omega\rangle\}} = \mathcal{H}_\omega.$$

$|\Omega_\omega\rangle$ is often called the "vacuum" vector. The expectation value of an operator $x \in \mathfrak{A}$ is given by the formula

$$\omega(x) = \langle\Omega_\omega|\pi_\omega(x)|\Omega_\omega\rangle.$$

It is not surprising that one cannot reduce the description of a C^*-dynamical system corresponding to time-evolution of our quantal system to this representation. In many concrete applications $|\Omega_\omega\rangle$ plays the rôle of a time-invariant "equilibrium", but a general system can exhibit phase transitions like magnetization in the case of the Ising spin model or Bose condensation in the example of a Bose gas. In general the situation is even worse and the time-evolution in a given representation does not exist for any small time interval. A related phenomenon is encountered in Example 1. Thus we do not expect the time-evolution to exist as a continuous (in some topology) 1-parameter group (or 1-parameter semigroup in the case of irreversible process) of $*$-automorphisms

$$\tau_\omega : \mathbb{R} \to \text{*-Aut } \mathcal{L}(\mathcal{H}_\omega).$$

There is another problem related to the non-existence of a global unitary time-evolution. Phase transitions should correspond to shifts of some "macroscopic" observables. But so far we do not account for any "macroscopic" observables in our approach. These are not elements of $\mathfrak{A}$ in general. In fact in many applications the centre of $\mathfrak{A}$ is trivial. These two problems are deeply related.

Consider a resolution of the problem of non-existence of the unitary time-evolution. Instead of working with one representation one can work on the level of C^*-algebra $\mathfrak{A}$ and consider all possible representations. On the level of $\mathfrak{A}$ a time-evolution should correspond to the 1-parameter group

$$\tau : \mathbb{R} \to \text{*-Aut } \mathfrak{A}.$$

If we take the universal representation $(\mathcal{H}_u, \pi_u, |\Omega_u\rangle)$ defined as

$$\mathcal{H}_u := \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} \mathcal{H}_\omega, \quad \pi_u := \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} \pi_\omega, \quad |\Omega_u\rangle := \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} |\Omega_\omega\rangle$$

we could try to define the time-evolution as a 1-parameter group

$$\tau : \mathbb{R} \to \text{*-Aut } \mathcal{L}(\mathcal{H}_u).$$

This "solution" suffers from taking into account too many operators on the non-separable Hilbert space $\mathcal{H}_u$ which do not correspond to any sensible "physical" observables. We will futher restrict τ based on these considerations in the "mean field" models considered in this Chapter.

In general it is difficult to construct τ for a given physical situation in any of these descriptions - this is true even for time-evolution processes near the "equilibria". If $\mathfrak{A}$ arises from a "thermodynamic limit" of quantal systems with finite number of degrees of freedom with the systems indexed by an infinite set Π, then the obvious way to define τ is as a limit of local time-evolutions τ^I defined for any $I \subset \Pi$ in some topology

$$\tau = \text{some topology - } \lim_I \tau^I.$$

The local time-evolutions τ^I can be given by Hamiltonians Q^I. For long range interactions such as those considered in this Chapter the limit does not exist in norm-topology. In general there is no limit Hamiltonian Q^Π corresponding to the time-evolution of the "thermodynamic limit" (for example see Strocchi [**30**]). Nevertheless one can still obtain a limiting time-evolution τ using the algebraic approach. This was shown in a special class of mean field models in Bóna [**8**] and will be generalized in this Chapter. Even if Q^Π does not exist it is often the case that for a state $\omega \in \mathcal{S}(\mathfrak{A})$ there exists Q_ω such that τ is Hamiltonian in the GNS representation

$$\frac{d}{dt}\tau_t(\pi_\omega(x)) = i[Q_\omega, \tau_t(\pi_\omega(x))],$$

see Sewell [**29**].

The second problem mentioned above is that of a lack of "macroscopic" observables. This is tied to the problem of τ non-invariance of $\mathfrak{A}$ in many relevant quantal systems of infinite number of degrees of freedom (for example see Dubin, Sewell [**14**], Rieckers [**25**]). The resolution of this problem is to work in a certain sub C^*-algebra $\mathfrak{C}$ of the double dual $\mathfrak{A}^{**}$ containing $\mathfrak{A}$. Here we naturally identify $\mathfrak{A}$ with the embedding $\pi_u(\mathfrak{A})$. Thus we consider $\mathfrak{A}^{**}$ as the weak closure of $\pi_u(\mathfrak{A})$ in $\mathcal{L}(\mathcal{H}_u)$. This is in fact a von Neumann algebra. We therefore seek τ as a 1-parameter group of $*$-automorphisms of $\mathfrak{C}$.

A natural question is what is the "macroscopic" limit of the system and the dynamics. In general this question is a hard one and should be approached by numerical analysis. We will consider systems which are thermodynamic limits of systems of finite number of degrees of freedom which are described by a unitary representation of some connected locally compact Lie group G - the group of kinematic symmetries. This symmetry will simplify the problem to one which can be solved exactly. Furthermore we will consider systems with a large order of "symmetry" of their dynamics in the sense that each subsystem will interact with each other in the same "way". In particular the mean field models in this Chapter exhibit arbitrarily long range interaction which is "infinitely weak". This will become clear in Section 5.

It turns out that the finite subsystems can be descibed exactly as systems in an external field with a Schrödinger equation governing the time-evolution of the finite subsystem. Furthermore the external field evolves according to a classical Hamiltonian dynamics on $\mathfrak{g}^*$ the dual of the Lie algebra of the group G. In fact $\mathfrak{g}^*$ (or rather some submanifold) can be thought of as the mean field phase space and it arises from taking into account "macroscopic" observables as Cesaro means of local quantal observables.

This Chapter is organized as follows. In Section 2 we will discuss the proper framework for quantal systems with infinite number of degrees of freedom together with a Lie group G of "kinematic symmetries". We will call such systems "thermodynamic G-systems". While our treatment is very general two fundamental examples will be considered which correspond to the usual intuition about thermodynamic limits of quantal systems with finite number of degrees of freedom.

Section 3 deals with the "macroscopic" phase space on the dual of the Lie algebra $\mathfrak{g}$ of G. A crucial rôle is played by what we call a "mean field connection", a projector-valued measure E on $\mathfrak{g}^*$ with values in the centre of the double dual of the C^*-algebra of the "thermodynamic G-system". While the definition and proof of existence of such "mean field connections" are abstract the two fundamental examples furnish canonical examples which are highly relevant to rigorous treatment of quantal systems with infinite number of degrees of freedom.

Section 4 introduces the "correct" C^*-algebra of observables $\mathfrak{C}$ for any thermodynamic G-system with a connection E. In particular $\mathfrak{C}$ accounts for the emergent "macroscopic" observables whose abelian von Neumann algebra can be identified with the algebra of (bounded) continuous functions on some subset of $\mathfrak{g}^*$. This corresponds to the intuitive situation in the two fundamental examples that the "macroscopic" phase space emerges from the Cesaro means of the generators of the G-action. This "macroscopic" system is in those examples the mean field generated by the "thermodynamic G-system".

Section 5 shows that we can define rigorously the time evolution of the entire quantal system using a classical Hamiltonian Q which governs the dynamics of the mean field. Again the crucial rôle is that of the "mean field connection". In the fundamental examples this time-evolution coincides with the time-evolution defined by a canonically associated Hamiltonian operator on any local finite subsystem of the infinite quantal system.

Section 6 discusses some generalizations of this setup and some open problems.

2. Thermodynamic G-systems

In this Section we will define and give examples of the mathematical structures which will describe our mean field theories. We will start with special type of colimits in the category of C^*-algebras. These are often referred to as C^*-inductive limits.

Remark. One can consider rather general colimits in the category of C^*-algebras. This is postponed to some other work, since here it would rather obscure the simple underlying considerations. For the type described here see e.g. Bratteli, Robinson [**12**]. For general colimits see e.g. MacLane [**21**], Rordam, Larsen, Laustsen [**27**]. □

Let us consider a net $\{\mathfrak{A}^I\}_I$ of C^*-algebras indexed by subsets $I \subset \Pi$ where Π is an infinite countable directed set with the relation of inclusion. Furthermore assume that if $I \subset J$ then $\mathfrak{A}^I \subset \mathfrak{A}^J$ (in principle a general colimit will have arbitrary maps between the related C^*-algebras). The C^*-inductive limit is denoted

$$\mathfrak{A}^\Pi := \lim_{\vec{I}} \mathfrak{A}^I = \overline{\cup_I \mathfrak{A}^I}.$$

We will assume that the C^*-algebras $\mathfrak{A}^I$ are unital. It is often useful to assume that if $I \cap J = \emptyset$ that $[\mathfrak{A}^I, \mathfrak{A}^J] = 0$.

The index set Π should be thought of as an indexing set of subsystems of our quantal system. We assume that these subsystems have finite number of degrees of freedom. Thus $\mathfrak{A}^I$ is a C^*-algebra playing the rôle of local algebra of observables corresponding to degrees of freedom "supported" on $I \subset \Pi$. The C^*-inductive limit of the entire C^*-inductive system is to be thought of as the algebra of the large quantal system. However as was discussed in the previous Section we will need to enlarge this C^*-algebra. The proper setting for observables of infinite quantal systems is the double dual $\mathfrak{A}^{**}$. As we will see this (von Neumann) algebra will have highly-nontrivial center, in particular it will contain macroscopic observables as a C^*-subalgebra. We recall some standard properties of $\mathfrak{A}^{**}$. For more details see Bratteli, Robinson [**12**] and Megginson [**22**].

THEOREM 2. *(Goldstine) Let $\mathfrak{A}$ be a Banach space (not necessarily a C^*-algebra). The closed unit ball $B_\mathfrak{A} \subset \mathfrak{A}$ is w^*-dense in the closed unit ball $B_{\mathfrak{A}^{**}} \subset \mathfrak{A}^{**}$ when considered under the image of the canonical isometric embedding of $\mathfrak{A}$ into $\mathfrak{A}^{**}$, $\overline{B_\mathfrak{A}}^{w^*} = B_{\mathfrak{A}^{**}}$.*

PROPOSITION 3. *Let $(\mathcal{H}_u, \pi_u, |\Omega_u\rangle)$ be the universal representation of $\mathfrak{A}$. Then*

$$\pi_u(\mathfrak{A})'' \cong \overline{\pi_u(\mathfrak{A})}^w \cong \mathfrak{A}^{**}$$

as Banach spaces. In particular this allows to define multiplication and involution on $\mathfrak{A}^{**}$ so that it is a von Neumann algebra.

We will denote the set of normal states (ultra-weakly continuous states) on $\mathfrak{A}^{**}$ by $\mathcal{S}_*(\mathfrak{A}^{**})$. We will use frequently the following well known result.

PROPOSITION 4. *Let $\omega \in \mathcal{S}(\mathfrak{A})$ be a state. Then it extends to a unique normal state (by abuse of notation) $\omega \in \mathcal{S}_*(\mathfrak{A}^{**})$.* □

We are now ready to start considering QM of systems with infinite number of degrees of freedom.

DEFINITION 2. *Let G be a connected finite-dimensional locally compact Lie group. A pair $(\mathfrak{A}^\Pi, \sigma)$ where $\mathfrak{A}^\Pi$ is a C^*-algebra with unitarily inequivalent irreducible representations and σ is a *-homomorphism*

$$\sigma : G \to *\text{-}Aut\ \mathfrak{A}^\Pi$$

resp. its extension by virtue of $\sigma\big((\mathfrak{A}^\Pi)^{**}, (\mathfrak{A}^\Pi)^*\big)$-density of $\mathfrak{A}^\Pi$ in $(\mathfrak{A}^\Pi)^{**}$

$$\sigma : G \to *\text{-}Aut\ (\mathfrak{A}^\Pi)^{**}$$

is called a thermodynamic G-system.

REMARK. Let us explain the rôle of the Lie group G. We will introduce two fundamental constructions - Example 1 and Example 2. Both are built from copies of a QM system with finite number of degrees of freedom. Example 1 is suitable for treatment of spin systems such as the BCS spin model, see Bóna [**9**]. Example 2 is constructed for infinite systems of interacting particles (the interaction is a rather special one - infinitely weak and infinitely long-range). In both cases these copies are equipped with a w^*-continuous irreducible representation of G on the corresponding C^*-algebra $\mathfrak{A} = \mathcal{L}(\mathcal{H})$. In particular our examples will come equipped with a weakly continuous irreducible unitary representation U. Therefore

$$\overline{\{U(g), g \in G\}}^w = \mathcal{L}(\mathcal{H}).$$

In particular we will build the local Hamiltonians out of the generators X_β of 1-parameter subgroups $U(\exp t\beta)$, $\beta \in \mathfrak{g}$ defined by (16) below. □

REMARK. As was extensively argued it is the algebraic structure of QM that is of prime importance in extension to quantal systems of infinite number of degrees of freedom and so it is more natural to define our thermodynamic G-systems using σ. If $\sigma : G \to {}^*\text{-Aut}\,\mathfrak{A}$ where $\mathfrak{A}$ is the C^*-algebra of a finite subsystem $\mathfrak{A} = \mathcal{L}(\mathcal{H})$ we can define U with the use of the obvious formula

$$\sigma(g)(\pi(x)) = U(g)\pi(x)U(g)^*. \tag{15}$$

In general the possibility of existence of U depends on Hochschild cohomology of $\mathfrak{A}$.

The problem with (15) is that U is not really unitary in general as it is defined only up to phase. The group law is satisfied only as

$$U(gh) = B(g,h)U(g)U(h)$$

where $B : G \times G \to \mathbb{R}/\mathbb{Z}$ and $g, h \in G$. It is well known that projective representations can be lifted to unitary representations of the central extension ${}^BG$ corresponding to the short exact sequence of Lie groups

$$1 \to \mathbb{R}/\mathbb{Z} \to {}^BG \to G \to 1$$

so that as a set ${}^BG$ is $\mathbb{R}/\mathbb{Z} \times G$ and the multiplication is defined by

$$(z, g)(w, h) = (B(g,h)zw, gh).$$

It is easy to prove that all central extensions of G by $\mathbb{R}/\mathbb{Z}$ are classified by $H^2(G, \mathbb{R}/\mathbb{Z})$. Therefore if we pass from σ to U we should think of U either as a projective representation or what is more practical as a unitary representation of ${}^BG$ for a suitable B. This ambiguity will appear in the next Section when we will discuss the macroscopic mean fields and mean field connections. □

EXAMPLE 1. Let $\mathfrak{A} = \mathcal{L}(\mathcal{H})$ be a C^*-algebra corresponding to some quantal system. Let U be a norm-continuous unitary representation of a connected compact Lie group G on $\mathcal{H}$, $U : G \to \mathcal{L}(\mathcal{H})$. Its generators X_β, $\beta \in \mathfrak{g}$ defined by

$$U(\exp(t\beta)) = \exp(-itX_\beta) \tag{16}$$

are bounded selfadjoint operators on $\mathcal{H}$. Let Π be a countable set. One can take the complete tensor product space (von Neumann [**33**])

$$\mathcal{H}^\Pi := \otimes_{i\in\Pi}\mathcal{H}_i$$

where $\mathcal{H}_i$ are copies of $\mathcal{H}$ by unitary maps $u_i : \mathcal{H} \to \mathcal{H}_i$. For any finite $I \subset \Pi$ we have a unitary representation of G given by the tensor product of copies of the unitary representation U, that is on any product state $|\psi\rangle = \otimes_{i\in I}|\psi_i\rangle$, $|\psi_i\rangle \in \mathcal{H}_i$

$$U^I : G \to \mathcal{L}(\otimes_{i\in I}\mathcal{H}_i) \tag{17}$$

$$U^I(g)|\psi\rangle := \otimes_{i\in I} u_i U(g) u_i^{-1}|\psi_i\rangle.$$

These are also norm-continuous and their generators

$$U^I(\exp(t\beta)) = \exp(-itX^I_\beta)$$

are just sums of the generators of the copies

$$X^I_\beta := \sum_{i\in I} \text{id} \otimes \ldots \otimes u_i X_\beta u_i^{-1} \otimes \ldots \otimes \text{id}.$$

These correspond to "approximations" of the extensive observables of our system. Clearly these generators diverge in the limit $I \to \Pi$. This is reflected by the fact that the "limit" of U^I will be strongly (weakly) discontinuous. This is one of the main sources of various pathologies in the treatment of quantal systems with infinite number of degrees of freedom as thermodynamic limits of finite quantal subsystems. To see this consider the unitary representation U^Π acting on $\mathcal{H}^\Pi$. If $|\phi\rangle_\Pi := \otimes_{i\in\Pi}|\phi\rangle_i \in \mathcal{H}^\Pi$ is a tensor product state then U^Π is defined as

$$U^\Pi(g)|\phi\rangle_\Pi := \otimes_{i\in\Pi} u_i U(g) u_i^{-1}|\phi\rangle_i.$$

If $|\psi\rangle_\Pi \in \mathcal{H}^\Pi$ is a tensor product state $|\psi\rangle_\Pi := \otimes_{i\in I} u_i|\psi\rangle$ such that $|\psi\rangle \in \mathcal{H}$ is not an eigenvector of X_β for some $\beta \in \mathfrak{g}$ then for arbitrary small $t > 0$ we have

$$\prod_{i\in\Pi} |{}_\Pi\langle\psi|u_i \exp(-itX_\beta)u_i^{-1}|\psi\rangle_\Pi| = \prod_{i\in\Pi} |\langle\psi|\exp(-itX_\beta)|\psi\rangle| = 0$$

since by Cauchy-Schwarz inequality we have

$$|\langle\psi|\exp(-itX_\beta)|\psi\rangle| < 1.$$

This implies that U^Π is weakly discontinuous.

We have the commutation relations of bounded generators

$$[X^I_\beta, X^I_\gamma] = iX^I_{[\beta,\gamma]}. \tag{18}$$

Consider the Cesaro means of X_β defined by

$$X_{\beta I} := \frac{1}{|I|} X^I_\beta \tag{19}$$

for any $\beta \in \mathfrak{g}$ and any $I \subset \Pi$ finite. In the physical language these are the "approximations" to the intensive observables of our system. Correspondingly we have the commutation relations

$$[X_{\beta I}, X_{\gamma I}] = \frac{i}{|I|} X_{[\beta,\gamma]I}. \tag{20}$$

Unlike for the extensive X^I_β limits of $X_{\beta I}$ can exist in suitable topologies. It is clear that then the commutation relations (20) then imply that these limits are "macroscopic" in the sense that they become commuting operators.

The algebraic counterpart of this example is as follows. Let us denote $\mathfrak{A} = \mathcal{L}(\mathcal{H})$ and corresponding to the unitary maps $u_i : \mathcal{H} \to \mathcal{H}_i$ as above let

$$\mathfrak{A}_i := \mathcal{L}(\mathcal{H}_i) = u_i \mathcal{L}(\mathcal{H}) u_i^{-1}.$$

For any finite $I \subset \Pi$ we define

$$\mathfrak{A}^I := \otimes_{i\in I} \mathfrak{A}_i$$

and let us denote the colimit

$$\mathfrak{A}^\Pi := \varinjlim_i \mathfrak{A}_i.$$

The unitary representation (16) induces an action of G on $\mathfrak{A}$ by inner automorphisms as follows

$$\sigma : G \to {}^*\text{-Aut } \mathfrak{A},$$

$$\sigma(g)(x) := U(g)xU(g^{-1}).$$

There is a natural extension of this action to $\mathfrak{A}^\Pi$. If $x = \otimes_{i\in I} x_i$ where $I \subset \Pi$ is finite and $x_i \in \mathfrak{A}_i$ then we can define an extension of σ to an action on $\mathfrak{A}^I$ by the formula

$$\sigma^\Pi(g)(x) := \otimes_{i\in I} u_i U(g) x_i U(g)^{-1} u_i^{-1}.$$

and this action can be extended by density to $\mathfrak{A}^\Pi$ resp $(\mathfrak{A}^\Pi)^{**}$ by Goldstine's theorem. It is easy to see that we have

$$\sigma^\Pi(g)(x) = U^\Pi(g)xU^\Pi(g)^{-1}.$$

□

Example 2. Let the notation be as Example 1 but let G be a locally compact (in general non-compact) Lie group and U be a strongly (or weakly) continuous unitary representation of G on $\mathcal{H}$. Then the generators X_β given by (16) are unbounded self-adjoint operators on $\mathcal{H}$. As before one can define strongly (or weakly) continuous unitary representations U^I of G on $\otimes_{i\in I}\mathcal{H}_i$ in the usual way taking tensor product representation (17). However to make sense of the commutation relations (18) it is necessary to face the issue of unboundedness of $X_{\beta I}$. This issue will be more important later on when we build up local Hamiltonians from polynomials of $X_{\beta I}$. □

There is however a simple yet deep result of Gårding [**15**] which constructs a dense domain $\dot{\mathcal{D}}$ common for all the generators X_β of 1-parameter subrepresentations of U. Furthermore these generators satisfy $X_\beta \dot{\mathcal{D}} \subset \dot{\mathcal{D}}$ so it makes sense to define the algebra generated by them.

PROPOSITION 5. *Let G be a locally compact group and $U : G \to \mathcal{L}(\mathcal{H})$ be a strongly continuous unitary representation. Then*

$$\dot{\mathcal{D}} := \bigcap_{\beta \in \mathfrak{g}} \mathcal{D}(X_\beta)$$

is dense in $\mathcal{H}$ and $X_\beta \dot{\mathcal{D}} \subset \dot{\mathcal{D}}$ for all $\beta \in \mathfrak{g}$.

3. Mean field G-systems and mean field connections

Now we will describe the linear Poisson space $\mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G. The "macroscopic" mean field lives on $\mathfrak{g}^*$ or rather its submanifold. Let us note that in principle more general colimits of C^*-algebras with more elaborate structure of the local generators of the unitary representations will lead to non-linear phase spaces replacing that of $\mathfrak{g}^*$. We postpone this discussion to some later work.

We will treat $\mathfrak{g}^*$ as a smooth manifold with the tangent and cotangent bundle identified with the trivial bundles $T\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}^*$, $T^*\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}$. We will consider $\mathfrak{g}^*$ as a Poisson manifold with the Kirilov-Kostant-Souriau (KKS) Poisson bivector (e.g. Kirillov [**18**])

$$\pi_F(d_F f, d_F g) := F([d_F f, d_F g]) \tag{21}$$

where $F \in \mathfrak{g}^*$, $f, g \in C^\infty(\mathfrak{g}^*)$ and the above identifications.

Let $b \in Z^1(G, \mathfrak{g}^*)$ be a symplectic cocycle of G, that is a 1-cocycle of Lie group cohomology of G with values in the coadjoint representation $\mathfrak{g}^*$. This means that b is closed under the differential of the Lie group cohomology with values in $\mathfrak{g}^*$ with the coadjoint representation

$$b(gh) = \mathrm{Ad}^*(g)b(h) + b(g). \tag{22}$$

Its linearization at the identity element $e \in G$ is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g}^*$

$$T_e b(\beta_1)(\beta_2) = B(\beta_1, \beta_2).$$

Here B is a bivector $B \in \Gamma(\wedge^2 T\mathfrak{g}^*)$.

The proof of the following Proposition is easy and is omitted.

PROPOSITION 6. *If $b \in Z^1(G, \mathfrak{g}^*)$ is a symplectic cocycle of G then*

$$(\pi_B)_F(df, dg) := \pi_F(d_F f, d_F g) + B_F(df, dg) \tag{23}$$

where $F \in \mathfrak{g}^*$, $f, g \in C^\infty(\mathfrak{g}^*)$ and π is KKS (21) is a Poisson structure on $\mathfrak{g}^*$ and

$$B(\beta_1, [\beta_2, \beta_3]) + \textit{cycl.} = 0. \tag{24}$$

Equation (24) is necessary for π_B to satisfy $[\pi_B, \pi_B] = 0$ where the bracket is the Schouten bracket of polyvector fields. Furthermore there is a smooth action Ad^*_B of G on $\mathfrak{g}^*$ such that its orbits are maximal integral submanifolds of $\mathfrak{g}^*$ with respect to π_B. It is easy to prove that this action

$$\mathrm{Ad}^*_B(g)(F) := \mathrm{Ad}^*(g)F + b(g).$$

We will call Ad^*_B the "extended" coadjoint action. The name "extended" in the above considerations come from the central extension of the considered group G. The linearization of a 1-cocycle v on G with values in $\mathfrak{g}^*$ is by definition a map $\mathfrak{g} \to \mathfrak{g}^*$ which can be considered as element of $\wedge^2\mathfrak{g}^*$. We can treat it in this way as a 2-cocycle on $\mathfrak{g}$ with values in $\mathbb{R}$. It is not difficult to show that $H^2(\mathfrak{g}, \mathbb{R})$ is isomorphic with $H^1(\mathfrak{g}, \mathfrak{g}^*)$ which justifies this viewpoint. Since $H^2(\mathfrak{g}, \mathbb{R}) \cong H^2(G, \mathbb{R}/\mathbb{Z})$ we have that $H^1(\mathfrak{g}, \mathfrak{g}^*)$ classifies central extensions of G. It is possible to reinterpret the extending 1-cocycle B and the modified KKS structure π_B and modified coadjoint action Ad^*_B on a given coadjoint orbit of G as the usual KKS structure and coadjoint action but with respect to the coadjoint orbit of the central extension ${}^B G$. More details can be found in Libermann, Marle [**20**]. As was discussed in the previous Section this is the correct setting if we wish to define the dynamics of $\mathfrak{A}^\Pi$ in terms of generators of U induced by σ.

DEFINITION 3. *A pair $(\mathfrak{g}^*, Ad^*_B)$ where $B \in \Gamma(\wedge^2 T\mathfrak{g}^*)$ satisfies (24) is called a* mean field G-system*.*

REMARK. In both examples we will construct we have $B = 0$. But the formalism introduced above is useful for more elaborate mean field models. In principle we will be proving theorems with $B = 0$ but the proofs are easily modified for nontrivial B. □

The mean field constructions will involve "quantizing" a mean field system. We should note that by this we do not mean the usual procedure of geometric or deformation quantization. Instead in our general setup we will already have both the infinite quantum system and classical mean field systems defined and there will be a "connection" which will allow us to transfer dynamics from the mean field to the quantal system and vice versa for a large class of Hamiltonians.

There is a beautiful connection between the classical and quantum world which is due to Kirillov [**18**] and it originates in what was originally a mathematical question to find all equivalence classes of irreducible unitary representations of a Lie group G. This is the "orbit method" which briefly states that for nilpotent compact Lie groups we have a bijection between the symplectic leaves of $\mathfrak{g}^*$ and equivalence classes of irreducible unitary representations. For more general Lie groups the method has its caveats and is still not well understood completely. We will not discuss the relations between this "mean field" quantization and the "orbit method" as these are still work in progress.

DEFINITION 4. *Let $(\mathfrak{A}^\Pi, \sigma)$ be a thermodynamic G-system and $(\mathfrak{g}^*, Ad^*_B)$ be a mean field G-system and $\mathcal{B}(\mathfrak{g}^*)$ the Borel σ-algebra of $\mathfrak{g}^*$. A mean field connection between $(\mathfrak{A}^\Pi, \sigma)$ and $(\mathfrak{g}^*, B)$ is a projector valued measure E on $\mathfrak{g}^*$ with values in $\mathcal{Z}((\mathfrak{A}^\Pi)^{**})$*

$$E : \mathcal{B}(\mathfrak{g}^*) \to \mathcal{Z}((\mathfrak{A}^\Pi)^{**})$$

such that E is G-equivariant

$$E(Ad^*_B(g)U) = \sigma(g)E(U).$$

A support of mean field connection E is the smallest closed Borel subset supp $E \subset \mathfrak{g}^*$ which satisfies

$$p_G := E(supp\ E) = E(\mathfrak{g}^*).$$

The dimension of an orbit of $F \in \mathfrak{g}^*$ is denoted as dim $\mathcal{O}_F$. By definition it is a non-negative even number. The dimension of a mean field connection E is

$$dim\ E := max\{dim\ \mathcal{O}_F,\ F \in supp\ E\}.$$

If dim $E = 0$ then the mean field connection is said to be trivial.

If for every $x \in (\mathfrak{A}^\Pi)$ and $\omega \in p_G\mathcal{S}(\mathfrak{A}^\Pi)$ the map $g \mapsto \omega(\sigma(g)x)$ is continuous E is said to be $\sigma(\mathfrak{A}^\Pi, p_G\mathcal{S}(\mathfrak{A}^\Pi))$-continuous.

A proof of existence of mean field connections can be found in Bóna [**10**]. Obviously, trivial mean field connections are not interesting. Example 1 and Example 2 of the previous Section have a natural non-trivial mean field connection given by explicit constructions.

EXAMPLE 3. Let the thermodynamic G-system be that of Example 1 and the mean field G-system be without extension $B = 0$. As noted the zero extension is because we construct σ from the norm-continuous unitary representation U and by Stone's Theorem we have

$$U(\exp(t\beta)) =: \exp(-itX_\beta) \in \mathcal{L}(\mathcal{H})$$

where X_β, $\beta \in \mathfrak{g}$ are bounded self-adjoint operators. Let p_G be the supremum of all projectors $p \in \mathcal{Z}(\mathfrak{A}^{**})$ such that the limits

$$X_{\beta\Pi} := \mathrm{s} - \lim_I X_{\beta I} p$$

exists for each $\beta \in \mathfrak{g}$ in $\mathcal{L}(\mathcal{H}_u)$. Then

$$U_\Pi : \mathfrak{g} \to \mathcal{L}(p_G\mathcal{H}_u),$$

$$U_\Pi(\beta) = \exp(iX_{\beta\Pi})$$

is a norm continuous unitary representation of the abelian locally compact group $\mathfrak{g}$. Let E be the unique projector valued measure on $\hat{\mathfrak{g}}$, the dual group of $\mathfrak{g}$ by the SNAG theorem (e.g. Bratteli, Robinson [**13**]). We can identify $\mathfrak{g}^*$ with $\hat{\mathfrak{g}}$ by the bijection taking $F \in \mathfrak{g}^*$ to the character χ_F

given by $\chi_F(\beta) := \exp(i(F(\beta))$ for any $\beta \in \mathfrak{g}$. Thus E is also a projector valued measure on $\mathfrak{g}^*$ with the property

$$X_{\beta\Pi} = \int_{\mathfrak{g}^*} F(\beta)E(dF). \tag{25}$$

E is a mean field connection as indeed we have G-invariance

$$\sigma(g)(E(U)) = E(\mathrm{Ad}^*(g)U)$$

for any $U \in \mathcal{B}(\mathfrak{g}^*)$. □

Example 4. Let the thermodynamic G-system $(\mathfrak{A}^\Pi, \sigma)$ be that of Example 2 and the mean field G-system be without extension $B = 0$ again since we construct σ from U. In this case U is strongly (weakly) continuous so X_β are unbounded self-adjoint operators on $\mathcal{H}$. One can still take the supremum p_G of all projectors $p \in \mathcal{Z}(\mathfrak{A}^{**})$ satisfying the following conditions.

- The 1-parameter unitary group in $\mathcal{L}(p\mathcal{H}_u)$
$$U_{\beta I}(t) = \exp(-itX^I_\beta)p$$
is a strongly continuous for all $\beta \in \mathfrak{g}$ and all finite $I \subset \Pi$.
- The thermodynamic limits $U_{\beta\Pi}$ exists for all $\beta \in \mathfrak{g}$ in the w^*-topology
$$U_{\beta\Pi}(t) := \sigma(\mathfrak{A}^{**}, \mathfrak{A}^*) - \lim_I \exp(-itX^I_\beta)p$$
- $U_{\beta\Pi}$ are strongly continuous in $\mathcal{L}(p\mathcal{H}_u)$ for all $\beta \in \mathfrak{g}$.

Then one can apply SNAG Theorem again. Now

$$U_\Pi : \mathfrak{g} \to \mathcal{L}(p_G\mathcal{H}_u)$$
$$U_\Pi(\beta) := U_{\beta\Pi}(1)$$

will be strongly continuous unitary representation of $\mathfrak{g}$ in $p_G\mathcal{H}_u$. In this case after identifying the dual group $\hat{\mathfrak{g}}$ with $\mathfrak{g}^*$ as in the previous Example we have that for all $\omega \in \mathcal{S}_*(\mathfrak{A}^{**})$ with $\omega(p_G) = 1$ and all $x, y \in \mathfrak{A}^{**}$

$$\omega(x\ U_{\beta\Pi}(1)\ y) = \int_{\mathfrak{g}^*} \exp(-iF(\beta))\omega(xE(dF)y).$$ □

As seen in these examples if the unitary G-representations on the finite subsystems $I \subset \Pi$ are not norm continuous the corresponding generators $X_{\beta I}$ are unbounded self-adjoint operators for each $\beta \in \mathfrak{g}$. Thus one cannot naively take their limit in some operator topology to define the thermodynamic limit generators $X_{\beta\Pi}$ up to some projector. However one can use the notion of limits in the resolvent sense, c.f. Reed, Simon [**23**]. We will denote the resolvent of x at $\lambda \notin \sigma(x)$ by $R_\lambda(x)$.

Definition 5. *A sequence $\{x_n\}$ of (unbounded) self-adjoint operators on $\mathcal{H}$ is convergent to x in the strong resolvent sense iff*

$$s - \lim_n R_\lambda(x_n) = R_\lambda(x)$$

for all $\lambda \in \mathbb{C}$ with Im $\lambda \neq 0$. We will denote

$$s_R - \lim_n x_n = x.$$

Remark. One defines analogously convergence in the norm or weak resolvent sense. One easily sees that convergence in weak resolvent sense equals convergence in strong resolvent sense by the following equality

$$\lim_n \|\big(R_\lambda(x_n) - R_\lambda(x)\big)|\psi\rangle\|^2 =$$
$$= \lim_n \Big(\langle R_{\bar\lambda}(x)R_\lambda(x)\psi|\psi\rangle - \langle R_\lambda(x_n)\psi|R_\lambda(x)|\psi\rangle - \langle R_\lambda(x)\psi|R_\lambda(x_n)|\psi\rangle + \langle R_\lambda(x)\psi|R_\lambda(x)|\psi\rangle\Big) = 0.$$

Thus the equivalence of strong and weak continuity of U in our examples will translate to equivalence of the corresponding resolvent sense convergences. □

EXAMPLE 4 (continued). Let $p_G \in \mathcal{Z}((\mathfrak{A}^{\Pi})^{**})$ as before. One can write by Stone's theorem

$$U_{\beta\Pi}(t) = \exp(-itX_{\beta\Pi})p_G$$

where $X_{\beta\Pi}$ is an unbounded selfadjoint operator in $\mathcal{L}(p_G\mathcal{H}_u)$. Then

$$X_{\beta\Pi} = s_R - \lim_I X_{\beta I}.$$

Indeed we have $U_{\beta\Pi}(t) = \mathrm{s} - \lim_I U_{\beta I}(t)p_G$ for any t. Let $\mu \in \mathbb{C}$ with Im $\mu < 0$ so that using the spectral measure $E_{\beta\Pi}$ of $X_{\beta\Pi}$

$$\begin{aligned}\big\langle R_\mu(X_{\beta\Pi})\psi|\phi\big\rangle &= \int_{\mathbb{R}} \langle E_{\beta\Pi}(d\lambda)\psi|\phi\rangle(\mu-\lambda)^{-1}\\ &= i\int_{\mathbb{R}} \langle E_{\beta\Pi}(d\lambda)\psi|\phi\rangle \int_0^\infty dt\ \exp(-it(\mu-\lambda)) = i\int_0^\infty dt\ \exp(-it\mu)\big\langle \exp(itX_{\beta\Pi})\psi|\phi\big\rangle.\end{aligned}$$

Therefore

$$R_\mu(X_{\beta\Pi})|\psi\rangle = i\int_0^\infty dt\ \exp(-it\mu)\exp(itX_{\beta\Pi})|\psi\rangle.$$

Strong convergence of $U_{\beta I}p_G$ gives

$$\begin{aligned}&\lim_I \|\big(R_\mu(X_{\beta I}) - R_\mu(X_{\beta\Pi})\big)|\psi\rangle\|\\ &\qquad\le \lim_I \int_0^\infty dt\ \exp(t\ \mathrm{Im}\ \mu)\|\big(\exp(itX_{\beta I}) - \exp(itX_{\beta\Pi})\big)|\psi\rangle\| = 0.\end{aligned}$$

It can be easily proven that if $R_\mu(X_{\beta I}) \to R_\mu(X_{\beta\Pi})$ strongly for one value $\mu \in \mathbb{C}$, Im $\mu \neq 0$ then it is true for all μ using the usual analytic power series expression for the resolvent. So finally $X_{\beta\Pi}$ is a strong resolvent sense limit of $X_{\beta I}$. □

REMARK. Let us note that equation (25) makes sense also for unbounded generators using functional calculus for unbounded functions, e.g. Birman, Solomjak [**3**]. □

4. C^*-algebra $\mathfrak{C}$ of observables of a thermodynamic G-mean field system

A mean field connection E between a thermodynamic G-system $(\mathfrak{A}^{\Pi}, \sigma)$ and a G-mean field system $(\mathfrak{g}^*, \mathrm{Ad}^*_B)$ allows us to rigorously define the algebra of "G-macroscopic" observables which are in general not elements of $\mathfrak{A}^{\Pi}$. An example of such "macroscopic" observables are the limits of (19) for $I \to \Pi$ of Example 1. These are the intensive "molar observables" obtained from the symmetry generators X_β by Cesaro means. The notion of a mean field connection E effectively solves both the problem of emergence of classical "macroscopic" observables and the problem of defining consistent time-evolution of systems with infinite number of degrees of freedom. This time-evolution is defined using a 1-parameter group of *-automorphisms of a certain C^*-subalgebra of $(\mathfrak{A}^{\Pi})^{**}$ which is constructed using a mean field connection specified by the given system. This is necessary since as discussed in the Introduction a general time-evolution does not leave $\mathfrak{A}^{\Pi}$ invariant. This corresponds to the necessity of working with the universal representation (or rather weak closure in $(\mathfrak{A}^{\Pi})^{**}$) of $\mathfrak{A}^{\Pi}$ because of the existence of unitarily inequivalent irreducible representations of $\mathfrak{A}^{\Pi}$. Thus the problem of emergence of classical "macroscopic" observables in the thermodynamic limit is tied to the problem of non-invariance of $\mathfrak{A}^{\Pi}$ under sensible time-evolutions.

Let G be a locally compact group, $(\mathfrak{A}^{\Pi}, \sigma)$ a thermodynamic G-system, $(\mathfrak{g}^*, \mathrm{Ad}^*_B)$ a mean field G-system and E be a mean field connection between $(\mathfrak{A}^{\Pi}, \sigma)$ and $(\mathfrak{g}^*, \mathrm{Ad}^*_B)$. Let $p_G := E(\mathfrak{g}^*) \in \mathcal{Z}((\mathfrak{A}^{\Pi})^{**})$. From this Section we will assume that $\mathfrak{A}^{\Pi}$ is simple. This will simplify Theorem 3.

DEFINITION 6. *The σ-invariant abelian von Neumann subalgebra $\mathfrak{N}$ of $\mathcal{Z}((\mathfrak{A}^{\Pi})^{**})$ generated by id $\in (\mathfrak{A}^{\Pi})^{**}$ and $E(U)$ for all $U \in \mathcal{B}(\mathfrak{g}^*)$ is called* the algebra of G-macroscopic observables of the mean field connection E. *It can be realized (by the Gelfand-Naimark Theorem, e.g* [**30**]*) as the algebra of continuous functions on its compact Hausdorff spectrum denoted Spec $\mathfrak{N}$. Let*

$$p_{\mathfrak{N}} : \mathcal{S}(\mathfrak{A}^{\Pi}) \to \mathcal{S}_*(\mathfrak{N}) \tag{26}$$

be the map given by the composition of the normal state extension map $\mathcal{S}(\mathfrak{A}^{\Pi}) \to \mathcal{S}_*((\mathfrak{A}^{\Pi})^{**})$ of Proposition 4 with the restriction to $\mathfrak{N}$.

REMARK. In the case of Examples 1 and 2 the algebra of G-macroscopic observables $\mathfrak{N}$ can be equivalently described as the von Neumann subalgebra of $\mathcal{Z}((\mathfrak{A}^\Pi)^{**})$ generated by id $\in (\mathfrak{A}^\Pi)^{**}$ and the spectral projectors in $\mathcal{L}(p_G\mathcal{H}_u)$ of $X_{\beta\Pi}$ for all $\beta \in \mathfrak{g}$. □

The state $p_\mathfrak{N}(\omega)$ is in a sense a "macroscopic limit" of a state $\omega \in \mathcal{S}(\mathfrak{A}^\Pi)$. This state can be represented as a Radon measure on Spec $\mathfrak{N}$.

Let us now review some properties of the abelian von Neumann algebra $\mathfrak{N}$ and the map $p_\mathfrak{N}$ (26). These properties were proven first in Bóna [**10**]. The following Proposition is straightforward and the proof is omitted.

PROPOSITION 7. *The map $p_\mathfrak{N}$ is $\sigma((\mathfrak{A}^\Pi)^*, (\mathfrak{A}^\Pi)^{**}) - \sigma(\mathfrak{N}^*, \mathfrak{N})$-continuous, affine, surjective and G-equivariant.*

The following Proposition will be crucial for the definition of the time-evolution of a thermodynamic G-system. It allows us to canonically decompose any state $\omega \in \mathcal{S}(\mathfrak{A}^\Pi)$ resp. its normal extension into states corresponding to points of Spec $\mathfrak{N}$. This is an interesting consequence of rigorous treatment of quantal systems with infinite number of degrees of freedom and is in sharp contrast with the fact that states do not form a simplex in QM. In particular to a normal state we associate a density matrix which we can decompose into convex combination of 1-dimensional projectors which are extreme points of the convex state of states. But this decomposition is highly non-unique unless the state is pure. Here the rôle of mean field connection is again essential since it gives rise to $\mathfrak{N}$ which in turn enables us to obtain the canonical decomposition of any state in terms of states on $\mathfrak{N}$.

PROPOSITION 8. *Let $\omega \in \mathcal{S}(\mathfrak{A}^\Pi)$ and denote μ_ω the canonical Radon measure on Spec $\mathfrak{N}$ corresponding to $p_\mathfrak{N}\omega$ obtained from the embedding of $\mathcal{S}_*(\mathfrak{N})$ into the dual of $C(\text{Spec } \mathfrak{N})$. Then there exists a canonical map*

$$\rho_\omega : \text{Spec } \mathfrak{N} \to \mathcal{S}((\mathfrak{A}^\Pi)^{**})$$

which is μ_ω-measurable and for any $x \in \mathfrak{A}^\Pi$ we have the canonical decomposition

$$\omega(x) = \int_{\text{Spec } \mathfrak{N}} \rho_\omega(m)(x)\mu_\omega(dm). \tag{27}$$

PROOF. The result will follow using decomposition theory which we sketch here. More details can be found in Bratteli-Robinson [**12**]. As usual we will consider $\omega \in \mathcal{S}(\mathfrak{A}^\Pi)$ as its extension to a normal state $\omega \in \mathcal{S}_*((\mathfrak{A}^\Pi)^{**})$. Let $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ be the corresponding GNS representation of $(\mathfrak{A}^\Pi)^{**}$. Let $M_\omega(\mathcal{S}((\mathfrak{A}^\Pi)^{**}))$ be the set of all positive Radon measures on the set of all states on $(\mathfrak{A}^\Pi)^{**}$ with barycenter ω, that is $\mu \in M_\omega(\mathcal{S}((\mathfrak{A}^\Pi)^{**}))$ iff

$$\omega = \int_{\mathcal{S}((\mathfrak{A}^\Pi)^{**})} \omega' \mu(d\omega').$$

There is a bijection between orthogonal measures $\mu \in M_\omega(\mathcal{S}((\mathfrak{A}^\Pi)^{**}))$, abelian von Neumann subalgebras $\mathfrak{M}_\mu \subset \pi_\omega((\mathfrak{A}^\Pi)^{**})'$ and orthogonal projections P_μ on $\mathcal{H}_\omega$ such that $P_\mu|\Omega_\omega\rangle = |\Omega_\omega\rangle$ and

$$P_\mu \pi_\omega((\mathfrak{A}^\Pi)^{**})P_\mu = \left(P_\mu \pi_\omega((\mathfrak{A}^\Pi)^{**})P_\mu\right)'.$$

Furthermore the given abelian von Neumann subalgebra is *-isomorphic to the range of the map

$$\kappa_\mu : L^\infty(\mu) \to \pi_\omega((\mathfrak{A}^\Pi)^{**})'$$

defined by

$$\langle \Omega_\omega | \kappa_\mu(f) \pi_\omega(x) | \Omega_\omega \rangle = \int_{\mathcal{S}((\mathfrak{A}^\Pi)^{**})} f(\omega')\omega'(x)\mu(d\omega')$$

for $x \in (\mathfrak{A}^\Pi)^{**}$.

Applying this result for the abelian von Neumann subalgebra $\pi_\omega(\mathfrak{N}) \subset \pi_\omega((\mathfrak{A}^\Pi)^{**})'$ there is an orthogonal measure $\mu_{\mathfrak{N},\omega} \in M_\omega(\mathcal{S}((\mathfrak{A}^\Pi)^{**}))$ which decomposes ω

$$\omega(x) = \int_{\mathcal{S}((\mathfrak{A}^\Pi)^{**})} \omega'(x)\mu_{\mathfrak{N},\omega}(d\omega')$$

for any $x \in (\mathfrak{A}^\Pi)^{**}$. We will show that this decomposition is exactly (27).

Let $p_\omega \in \mathfrak{N}$ be the unique projector such that $p_\omega \mathfrak{N} \cong \pi_\omega(\mathfrak{N})$. Then the above result states that $p_\omega \mathfrak{N} \cong L^\infty(\mu_\omega)$. This *-isomorphism is realized by

$$\varphi : p_\omega \mathfrak{N} \to L^\infty(\mu_\omega)$$
$$\varphi(x)(\omega') = \omega'(x).$$

Let $i : \mathfrak{N} \to (\mathfrak{A}^\Pi)^{**}$ be the inclusion. Since $\omega'(x) = 0$ for any $x \in (\mathrm{id} - p_\omega)\mathfrak{N}$ and $\omega' \in \mathrm{supp}\ \mu_\omega$ we have that $i^*\omega'$ is a pure state on $\mathfrak{N}$. Denote the corresponding point m' in Spec $\mathfrak{N}$. Using this notation let $\rho_\omega(m') := \omega'$ where $i^*\omega'$ corresponds to m'. It is easy to see that $i^*|_{\mathrm{supp}\ \mu_\omega}$ gives a bijection onto supp p_ω. Furthermore $p_{\mathfrak{N}}\omega = i^*\omega$ so μ_ω in (27) is

$$\mu_\omega(U) = \mu_{\mathfrak{N},\omega}((i^*)^{-1}(U))$$

for any $U \in \mathcal{B}(\mathcal{S}((\mathfrak{A}^\Pi)^{**}))$. □

It remains to define the C^*-algebra $\mathfrak{C}$ which involves both the local quantal observables as well as the macroscopic observables. The following topology will be important for the construction of $\mathfrak{C}$.

Definition 7. *The s^*-topology on $\mathfrak{A}^\Pi$ is a locally convex topology generated by seminorms*

$$q_\omega(x) := \omega(x^*x)^{1/2}, \qquad q^*_\omega(x) := \omega(xx^*)^{1/2}$$

where $\omega \in \mathcal{S}(p_G(\mathfrak{A}^\Pi)^{**})$.

Theorem 3. *The set $\mathfrak{C}$ of all uniformly bounded s^*-continuous functions $\hat{f} : supp\ E \to \mathfrak{A}^\Pi$ is a C^*-algebra with the naturally defined operations induced by the C^*-algebra operations of $\mathfrak{A}^\Pi$. Furthermore the mean field connection E defines a map (for compact supp E)*

$$E : \mathfrak{C} \to p_G(\mathfrak{A}^\Pi)^{**}$$

$$E : \hat{f} \mapsto E(\hat{f}) := \int_{supp\ E} \hat{f}(F) E(dF) \tag{28}$$

resp. for any supp E a map given by the formula using any $\omega \in p_G\mathcal{S}_*(\mathfrak{A}^{**})$ and the decomposition (27) as follows

$$\omega(E(\hat{f})) := \lim_{bounded\ B \subset \mathfrak{g}^*} \int_{Spec\ \mathfrak{N}} \rho_\omega(m)(E(B)\hat{f}(F_{\mathfrak{N}}(m)))\mu_\omega(dm) \tag{29}$$

and E is a C^*-isomorphism onto a C^*-subalgebra of $p_G(\mathfrak{A}^\Pi)^{**}$. Here $F_{\mathfrak{N}}(m) \in \mathfrak{g}^*$ with $m \in Spec\ \mathfrak{N}$ is defined as

$$F_{\mathfrak{N}}(m)(\beta) := m\Big(\int_B F(\beta)E(dF)\Big) \tag{30}$$

if $B \subset \mathfrak{g}^*$ is a bounded Borel subset such that $m(E(B)) = 1$.

Remark. The definition (30) is independent of B with $m(E(B)) = 1$. However $F_{\mathfrak{N}}(m)$ should be considered as a function on a 1-point compactified $\mathfrak{g}^*$ as (30) blows up if there is no bounded Borel $B \subset \mathfrak{g}^*$ with $m(E(B)) = 1$. Furthermore since for some $m \in \mathrm{Spec}\ \mathfrak{N}$ we have $m(\mathrm{id} - p_G) = 1$ one should further add an isolated point (the value of $F_{\mathfrak{N}}(m)$) to $\mathfrak{g}^*$. In the proof and in further considerations we shall abuse notation and write $F_{\mathfrak{N}}(m) \in \mathfrak{g}^*$. □

Proof. Completeness of $\mathfrak{C}$ follows from the usual argument for function spaces with values in a Banach space. Let $\{\hat{f}_n\}$ be a Cauchy sequence in $\mathfrak{C}$. This implies that for every $F \in \mathrm{supp}\ E$ the sequence $\{\hat{f}_n(F)\}$ is Cauchy sequence in $\mathfrak{A}^\Pi$ and thus there exists norm limits

$$\hat{f}(F) = \lim_n \hat{f}_n(F)$$

for each $F \in \mathrm{supp}\ E$. One needs to prove first that $\hat{f} \in \mathfrak{C}$. Let $\omega \in p_G\mathcal{S}(\mathfrak{A}^\Pi)$, $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ be the corresponding GNS representation and let $F_k \to F$ so that

$$q_\omega\Big(\hat{f}(F_k) - \hat{f}(F)\Big) = \|\big(\hat{f}(F_k) - \hat{f}(F)\big)|\Omega_\omega\rangle\| \le 2\|\hat{f}_n - \hat{f}\| + \|\big(\hat{f}_n(F_k) - \hat{f}_n(F)\big)|\Omega_\omega\rangle\|$$

$$= 2\|\hat{f}_n - \hat{f}\| + q_\omega\big(\hat{f}_n(F_k) - \hat{f}_n(F)\big) \to 0.$$

Similarily one can do the same calculation for the seminorm q^*_ω so that $\hat{f} \in \mathfrak{C}$. Finally we can finish the proof of completeness since for $n \to \infty$

$$\|\hat{f}_n - \hat{f}\| = \sup_{F \in \text{ supp } E} \|\hat{f}_n(F) - \hat{f}(F)\| = \lim_m \|\hat{f}_n(F) - \hat{f}_m(F)\| \to 0.$$

The C^*-property is proven straightforwardly

$$\|\hat{f}\|^2 = \Big(\sup_{F \in \text{ supp } E} \|\hat{f}(F)\|\Big)^2 = \sup_{F \in \text{ supp } E} \|\hat{f}(F)\|^2 = \sup_{F \in \text{ supp } E} \|\hat{f}(F)^*\hat{f}(F)\| = \|\hat{f}^*\hat{f}\|.$$

Now to prove the C^*-isomorphism we first need to prove that the integral formula (29) is well defined. This is easy in case of compact supp E where we can directly take (28) as definition. The general case is a lot harder. First for any $x \in (\mathfrak{A}^\Pi)^{**}$, $\omega \in p_G\mathcal{S}_*((\mathfrak{A}^\Pi)^{**})$ and $U \in \mathcal{B}(\text{Spec } \mathfrak{N})$ we have $\rho_\omega(m)(x)\chi_U(m)$ is Borel function on Spec $\mathfrak{N}$. $F_\mathfrak{N}$ given by (30) is continuous in $m \in$ Spec $\mathfrak{N}$ since the topology on Spec $\mathfrak{N}$ is w^*-topology. This implies that the integrand in (29) is a Borel function since we can use approximation of $F_\mathfrak{N}$ (as function on Spec $\mathfrak{N}$ with values in compactified $\mathfrak{g}^*$) by simple functions $\{F^n_\mathfrak{N}\}_n$ and use s^*-continuity of $\hat{f}$ to construct a sequence $\{\rho_\omega(m)(\hat{f}(F^n_\mathfrak{N}(m)))\}_n$ of uniformly bounded measurable functions which converge pointwise to the integrand in (29).

It is straightforward to verify that $E(\hat{f}) \in p_G(\mathfrak{A}^\Pi)^{**}$. Linearity of (28) is clear since in (29) $\rho_\omega(m)$ is linear for any $m \in$ Spec $\mathfrak{N}$. It is clear that by linearity we have using polarization

$$\omega(E(\hat{f})x) = \lim_{\text{bounded } B \subset \mathfrak{g}^*} \int_{\text{Spec } \mathfrak{N}} \rho_\omega(m)(E(B)\hat{f}(F_\mathfrak{N}(m))x)\mu_\omega(dm) \tag{31}$$

for any $x \in (\mathfrak{A}^\Pi)^{**}$ and $\hat{f} \in \mathfrak{C}$. We can apply this result for ω replaced by $\rho_\omega(m)$ so that

$$\rho_\omega(m)(xE(\hat{f})) = \rho_\omega(m)(x\hat{f}(F_\mathfrak{N}(m))). \tag{32}$$

Using (32) for $x = E(\hat{g})$ with $\hat{g} \in \mathfrak{C}$ in (31) we have that E is multiplicative. It is easy to see that E preserves involution.

The proof that the kernel of (28) is trivial is analogous to that of Bóna [**8**], for full details see also Bóna [**10**]. □

Remark. The non-compact case turned out to be a lot more difficult to work with. This is since except for fixed points non-compact Lie groups have usually non-compact coadjoint orbits so that supp E is unbounded. In particular for semi-simple non-compact Lie groups the only compact coadjoint orbit is the origin. □

Remark. We can identify the algebra of G-macroscopic observables $\mathfrak{N}$ with a C^*-subalgebra of $\mathfrak{C}$ of functions with values in λI, $\lambda \in \mathbb{C}$ and this C^*-subalgebra can be in the compact case identified with $C(\text{supp } E)$. □

Example 5. In Example 1 the C^*-algebra $\mathfrak{C}$ has a simple description as a tensor product

$$\mathfrak{C} \cong \mathfrak{A} \otimes \mathfrak{N}$$

see Bóna [**8**]. Furthermore the colimit structure of $\mathfrak{A}^\Pi$ defines a net $\{\mathfrak{C}^I\}$, $I \subset \Pi$ which gives $\mathfrak{C}$ the structure of a quasi-local C^*-algebra of observables as a colimit

$$\mathfrak{C} = \varinjlim_I \mathfrak{C}^I.$$

The restrictions of the map (28) to the subalgebras of $\mathfrak{C}$ given by all uniformly bounded s^*-continuous functions $\hat{f} : \text{supp } E \to \mathfrak{A}^I$ are C^*-isomorphisms onto $p_G(\mathfrak{A}^I)^{**}$.

This description has a nice physical interpretation since we have effectively replaced the net of local observables $\{\mathfrak{A}^I\}$ by a net $\{\mathfrak{C}^I\}$ which in addition to the local observables contains also intensive "macroscopic" observables which are elements of $\mathfrak{N}$. In fact the net $\{\mathfrak{C}^I\}$ is a net of tensor product algebras $\{\mathfrak{C}^I = \mathfrak{A}^I \otimes \mathfrak{N}\}$. This can be easily proven using the colimit structure e.g. see Blackadar [**4**] which suggests generalization to more general colimits.

In this example there is an additional property that one can treat $\mathfrak{N}$ as being independent on the local quantal systems. This is in accordance with its origin from thermodynamic limit. Let us prove this independence by showing that $\mathfrak{N}$ is G-invariant with respect to σ. This follows since by the definition of the thermodynamic limit generators $X_{\beta\Pi}$, $\beta \in \mathfrak{g}$ we have

$$\sigma(g)(X_{\beta\Pi}) = X_{\mathrm{Ad}(g)\beta\Pi}$$

for $g \in G$ so that

$$\sigma(g)(E(U)) = E(\mathrm{Ad}^*(g)U)$$

for Borel $U \subset \mathfrak{g}^*$ by G-equivariance of the G-mean field connection E. □

5. One-parameter groups of time-evolutions φ_t^Q and τ_t^Q

If G is a compact Lie group then for any mean field G-system $(\mathfrak{g}^*, \mathrm{Ad}_B^*)$ the symplectic leaves (orbits of the extended coadjoint action) are compact submanifolds. Any $Q \in C^\infty(\mathfrak{g}^*)$ defines a Hamiltonian vector field X_Q which is complete. This enables to define the Hamiltonian time-evolution flow φ_t^Q on $\mathfrak{g}^*$. In the case of a locally compact Lie group G this is no longer true and global time-evolution of a mean field G-system coming from a thermodynamic G-system is at stake. The coadjoint orbits are in general not compact and in fact compact coadjoint orbits of non-compact Lie groups are usually single point. However one has a rather broad class of functions $Q \in C^\infty(\mathfrak{g}^*)$ for which the flow φ_t^Q is guaranteed to exist at all times. This is an easy corrolary of a Lemma proven by Gordon [**16**]. Let us recall that a function is called proper iff the preimage of any compact set is compact.

Proposition 9. *Let $X_Q \in \Gamma(T\mathfrak{g}^*)$ be a vector field. If there exists a function f, a proper function g and constants C_1, C_2 such that*

$$|X_Q f(F)| \leq C_1 |f(F)|, \tag{33}$$

$$|g(F)| \leq C_2 |f(F)|, \tag{34}$$

for all points $F \in \mathfrak{g}^*$ then X_Q is complete. In particular if X_Q is a Hamiltonian vector field for a bounded from below proper $Q \in C^\infty(\mathfrak{g}^*)$ then X_Q is complete.

Proof. The derivative of f along integral curves is given by

$$X_Q f(F(t)) = \frac{d}{ds}\Big|_{s=0} f(F(t+s)).$$

By integration and (33) we have

$$|f(F(t))| \leq |f(F(0))| + \int_0^t dt' |X_Q f(F(t'))| \leq |f(F(0))| + C_1 \int_0^t dt' |f(F(t'))|$$

and Gronwall's lemma implies

$$|f(F(t))| \leq |f(F(0))| \exp(C_1 t).$$

By (34) we get the bound

$$|g(F(t))| \leq C_2 |f(F(0))| \exp(C_1 t).$$

This implies that $g(F(t))$ is bounded on any bounded time interval so since g is proper $F(t)$ is contained in a compact set and hence X_Q is complete.

The second part of the Proposition follows by setting $f = g = Q$ and $C_1 = 0, C_2 = 1$ since we have

$$|X_Q Q(F)| = |\{Q, Q\}(F)| = 0,$$

so that X_Q is complete. □

Remark. Note that Gordon's Lemma can be actually used for any (non-linear) Poisson manifold. This brings the possibility to generalize construction to thermodynamic G-systems with non-linear mean field G-system.

□

Given a bounded from below proper function $Q \in C^\infty(\mathfrak{g}^*)$ we can treat it as a Hamiltonian of a classical dynamical system on $\mathfrak{g}^*$ for a given symplectic cocycle b using the extended Poisson structure π_B on $\mathfrak{g}^*$. It leaves the extended coadjoint orbits (orbits of Ad_B^*) invariant. The following Proposition will show that it can be encoded equivalently into a certain function satisfying a cocycle property. Using this function one can define the time-evolution of both the "physically relevant" part of $\mathfrak{A}^\Pi$ and the "G-macroscopic" observables $\mathfrak{N}$ which are encoded in $\mathfrak{C}$.

Definition 8. *Let $(\mathfrak{g}^*, Ad_B^*)$ be a mean field G-system and $Q \in C^\infty(\mathfrak{g}^*)$ a bounded below proper function. A map*

$$g_Q : \mathbb{R} \times \mathfrak{g}^* \to G$$

satisfying

$$\frac{d}{dt}\Big|_{t=0} g_Q(t,F) = d_F Q, \qquad g_Q(0,F) = e,$$
$$g_Q(s, \varphi_t^Q(F)) g_Q(t,F) = g_Q(s+t, F)$$

for $F \in \mathfrak{g}^*$, $s,t \in \mathbb{R}$ is called a dynamical cocycle corresponding to Q.

Proposition 10. *Let $(\mathfrak{g}^*, Ad_B^*)$ be a mean field G-system. Let $Q \in C^\infty(\mathfrak{g}^*)$ be a bounded below proper function. Then there exists a unique dynamical cocycle g_Q corresponding to Q.*

Proof. Follows from Proposition 9 by completeness, for details see e.g. Varadarajan [**32**]. □

Remark. The dynamical cocycle g_Q can be viewed as a generator of the flow φ_t^Q since

$$\varphi_t^Q(F) = \mathrm{Ad}_B^*(g_Q(t,F))F \tag{35}$$

for all $t \in \mathbb{R}$ and all $F \in \mathfrak{g}^*$. □

Let us now turn to the dynamics of thermodynamic G-system $(\mathfrak{A}^\Pi, \sigma)$. Let G be a compact Lie group and consider $(\mathfrak{A}^\Pi, \sigma)$ as in Example 1. We will introduce dynamics for each finite $I \subset \Pi$ in the following way. Let $Q \in S(\mathfrak{g}^*)$ be a polynomial written in basis $\{\beta_1, \ldots, \beta_n\}$ of $\mathfrak{g}$

$$Q(\beta_1, \ldots, \beta_n) = \sum_{k=0}^{N} Q_k(\beta_1, \ldots, \beta_n)$$

where $Q_k \in S(\mathfrak{g}^*)$ are homogeneous polynomials. One can define after a suitable rearrangement a bounded self-adjoint operator

$$Q^I = |I| \sum_{k=0}^{N} Q_k(X_{\beta_1 I}, \ldots, X_{\beta_n I}) \tag{36}$$

where $X_{\beta_i I}$ are defined by (19). This suggest that we should treat a classical Hamiltonian Q as an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^*)$. Q^I will be the Hamiltonians of the finite subsystems with C^*-algebras $\mathfrak{A}^I$. The corresponding local time-evolutions given by one-parameter groups $\tau_t^{Q^I}$ of $*$-automorphisms of $\mathfrak{A}^I$ are given by inner *-automorphisms

$$\tau_t^{Q^I}(x) = \exp(itQ^I)\, x\, \exp(-itQ^I) \tag{37}$$

for $x \in \mathfrak{A}^I$.

In this Section we will construct time-evolution τ_t^Q of the C^*-algebra $\mathfrak{C}$ of Section 4 which will coincide with the local time-evolutions $\tau_t^{Q^I}$ on any finite $I \subset \Pi$. This time-evolution τ_t^Q can be defined rather generally for any thermodynamic G-system with a mean field connection. Without the colimit structure of $(\mathfrak{A}^\Pi, \sigma)$ the thermodynamic interpretation is meaningless - we simply define a time-evolution of a quantal systems with infinite degrees of freedom from considerations of the emergent "macroscopic" system.

In the case of Example 1 one can rigorously construct a limit of the local time-evolutions (37) in the s^*-topology given in Definition 7 as was shown in Bóna [**8**]. We will recall this in the next Section.

In Example 2 the Lie group G is locally compact and U^I are strongly (or weakly) continuous and $X_{\beta I}$ are unbounded selfadjoint operators for finite $I \subset \Pi$. We already mentioned the existence of Gårding domain in Proposition 5. Now further issue arises from the unboundedness of $X_{\beta I}$. Suppose we pick a classical Hamiltonian $Q \in \mathcal{U}(\mathfrak{g}^*)$ as above. Here ordering will be crucial. In general (36) does not define an essentially self-adjoint operator even after any rearrangement. There is however a large class of "physically" relevant classical Hamiltonians for which this is true. In particular for all practical purposes it is enough to restrict to the class of symmetric and elliptic elements of $\mathcal{U}(\mathfrak{g}^*)$, see Barut, Rączka [**2**]. Here symmetric means that it is invariant under the naturally defined antipode of $\mathcal{U}(\mathfrak{g}^*)$ and elliptic means that the corresponding partial differential operator on G is elliptic. Then (36) defines after suitable rearrangement essentially self-adjoint operators and we get the local time-evolutions defined by (37) for all $I \subset \Pi$. The issue of existence of s^*-limit of (37) is now more subtle and is postponed to the next Section.

Remark. In the bounded case one can construct the dynamics using any $Q \in C^\infty(\mathfrak{g}^*)$, see Bóna [**8**]. □

The above considerations lead us to the following definition. Notice that any polynomial is proper.

Definition 9. *An element $Q \in \mathcal{U}(\mathfrak{g}^*)$ is called an* admissible Hamiltonian *iff*

- *Q is bounded from below when considered as $Q \in C^\infty(\mathfrak{g}^*)$* (classical admissibility),
- *Q is symmetric and elliptic* (quantum admissibility).

Let $Q \in \mathcal{U}(\mathfrak{g}^*)$ be an admissible Hamiltonian. We will often treat it as an element $Q \in C^\infty(\mathfrak{g}^*)$. We will describe the idea of using the classical Hamiltonian flow φ_t^Q to define the time-evolution τ_t^Q of the thermodynamic G-system $(\mathfrak{A}^\Pi, \sigma)$ using a mean field connection E. τ_t^Q will be a one-parameter group of *-automorphisms of the C^*-algebra $\mathfrak{C}$. Let $\omega \in \mathcal{S}(\mathfrak{A}^\Pi)$ be a state and let ρ_ω be the map corresponding to the canonical decomposition (27) of Proposition 8. Using the mean field dynamics φ_t^Q we can define a time-evolution of the states $\rho_\omega(m)$ by the formula

$$\rho_\omega^t(m) := \sigma^*(g_Q(t, F_{\mathfrak{N}}(m)))\rho_\omega(m) \tag{38}$$

here $F_{\mathfrak{N}}(m) \in \mathfrak{g}^*$ is the corresponding element of $\mathfrak{g}^*$ corresponding to the classical delta-measure on $\mathfrak{g}^*$ associated to $\rho_\omega(m)$. Since these delta-measures evolve by formula (35) as

$$\varphi_t^Q(F_{\mathfrak{N}}(m)) = \mathrm{Ad}^*(g_Q(t, F_{\mathfrak{N}}(m)))F_{\mathfrak{N}}(m)$$

the time-evolution (38) is well defined by G-equivariance of E. Since the decomposition (27) of Proposition 8 is canonical we can define unambiguously define the time-evolution of a state $\omega \in p_G\mathcal{S}(\mathfrak{A}^\Pi)$ by the formula

$$\omega(\tau_t^Q(x)) := \int_{\mathrm{Spec}\ \mathfrak{N}} \sigma^*(g_Q(t, F_{\mathfrak{N}}(m)))\rho_\omega(m)(x)\mu_\omega(dm)$$

where we consider ω extended to a normal state on $p_G(\mathfrak{A}^\Pi)^{**}$ as usual. We will prove later that this is indeed a well-defined one-parameter group τ_t^Q of *-automorphisms of the C^*-subalgebra $\mathfrak{C}$ of $p_G(\mathfrak{A}^\Pi)^{**}$.

Definition 10. *Let $(\mathfrak{A}^\Pi, \sigma)$ be a thermodynamic G-system and $(\mathfrak{g}^*, Ad_B^*)$ be a mean field G-system and*

$$E : \mathcal{B}(\mathfrak{g}^*) \to \mathcal{Z}((\mathfrak{A}^\Pi)^{**})$$

*be a non-trivial mean field connection. Let $Q \in \mathcal{U}(\mathfrak{g}^*)$ be an admissible Hamiltonian. Then the classical mean field time-evolution φ_t^Q determines a one-parameter group of *-automorphisms of the C^*-algebra $\mathfrak{C} = C_{bs}(supp\ E, \mathfrak{A}^\Pi)$ using the formula*

$$\tau_t^Q(\hat{f})(F) := \sigma(g_Q^{-1}(t, F))\hat{f}(\varphi_t^Q(F)). \tag{39}$$

We will often identify $\mathfrak{C}$ with its image under the mean field connection understood as a map $E : \mathfrak{C} \to p_G(\mathfrak{A}^\Pi)^{**}$ defined by (28) as in Theorem 3.

PROPOSITION 11. *For any admissible Hamiltonian $Q \in \mathcal{U}(\mathfrak{g}^*)$ the one-parameter group of *-automorphisms τ_t^Q defined by (39) is a $\sigma(\mathfrak{C}, p_G\mathcal{S}_*((\mathfrak{A}^\Pi)^{**}))$-continuous group.*

PROOF. We need to prove that

$$\lim_{t\to 0} \omega(\tau_t^Q(E(\hat{f}))) = \omega(E(\hat{f}))$$

for any fixed $\omega \in p_G\mathcal{S}_*((\mathfrak{A}^\Pi)^{**})$ and any fixed $\hat{f} \in \mathfrak{C}$. We will use the decomposition (27) corresponding to ω and the notation used there. The same argument as in the proof of Theorem 3 shows that

$$\rho_\omega(m)(\tau_t^Q(\hat{f}(F_\mathfrak{N}(m)))) = \rho_\omega(m)(\sigma(g_Q^{-1}(t, F_\mathfrak{N}(m)))(\hat{f}(\varphi_t^Q(F_\mathfrak{N}(m))))). \tag{40}$$

is a measurable function of m. Furthermore (40) is continuous in t. This is a consequence of both g_Q and the corresponding flow φ^Q being bicontinuous as functions on $\mathbb{R} \times \mathfrak{g}^*$. Therefore

$$\begin{aligned}(41)\quad & \lim_{t\to 0} \omega(\tau_t^Q(E(\hat{f})) \\ &= \lim_{t\to 0} \int_{\mathrm{Spec}\ \mathfrak{N}} \rho_\omega(m)(\tau_t^Q(\hat{f})(F_\mathfrak{N}(m)))\mu_\omega(dm) = \int_{\mathrm{Spec}\ \mathfrak{N}} \rho_\omega(m)(\hat{f}(F_\mathfrak{N}(m)))\mu_\omega(dm) = \omega(E(\hat{f}))\end{aligned}$$

where the Dominated Convergence Theorem was used with the bound

$$|\rho_\omega(m)(\tau_t^Q(\hat{f}(F_\mathfrak{N}(m))))| \leq \|\hat{f}\|$$

for any $m \in \mathrm{supp}\ \mu_\omega$ and any t which is a consequence of unit norm of $*-$automorphisms $\sigma(g)$. □

Recall that $\mathfrak{N}$ is the C^*-subalgebra of $\mathfrak{C}$ consisting of all functions $\hat{f}$ on supp E with values in scalar multiples of the identity operator and we naturally identify $\mathfrak{N}$ with the algebra of "classical" observables. The following Proposition states that these "classical" observables are left invariant by τ_t^Q.

PROPOSITION 12. *The C^*-algebra $\mathfrak{N} \subset \mathfrak{C} \subset p_G(\mathfrak{A}^\Pi)^{**}$ is τ_t^Q-invariant. Furthermore for any finite $I \subset \Pi$ the C^*-subalgebra $\mathfrak{C}^I \subset p_G(\mathfrak{A}^\Pi)^{**}$ is τ_t^Q-invariant.*

PROOF. Since σ acts trivially

$$\sigma(g)(\lambda\ \mathrm{id}_{\mathfrak{A}^\Pi}) = \lambda\ \mathrm{id}_{\mathfrak{A}^\Pi}$$

for $\lambda \in \mathbb{C}$ we have that the time-evolution τ_t^Q given by (39) reduces for $\hat{f} \in \mathfrak{N}$ to

$$\tau_t^Q(\hat{f})(F) = \hat{f}(\varphi_t^Q(F))$$

so that $\tau_t^Q(\hat{f}) \in \mathfrak{N}$. The τ_t^Q-invariance of $\mathfrak{C}^I$ for any finite $I \subset \Pi$ follows from σ-invariance of $\mathfrak{A}^I$. □

It remains to justify τ_t^Q as the time-evolution of the infinite quantal system. For this we will consider the tractable cases of Example 1 and 2. This will be done by proving that the time-evolution τ_t^Q coincides on $\mathfrak{A}^I$ for any finite $I \subset \Pi$ with local time-evolutions defined in the obvious way by the inner automorphisms using local Hamiltonians (36). Furthermore in the next Section we will recall the result of Bóna [**8**] that in the case of Example 1 a stronger statement is true, namely that the "thermodynamic limit" of the local time-evolutions is exactly τ_t^Q. That the same is true for Example 2 is conjectured.

We consider the more general case of Example 2. Let $Q \in \mathcal{U}(\mathfrak{g}^*)$ be an admissible Hamiltonian and let the local Hamiltonians

$$Q^I := |I|Q(X_{\beta_1 I}, \ldots, X_{\beta_n I})$$

be defined in a way so that they are essentialy self-adjoint. Then

$$\tau_t^{Q^I}(x) = \exp(itQ^I)\ x\ \exp(-itQ^I)$$

for any $x \in \mathfrak{A}^I$ defines a 1-parameter group of *-automorphisms of $\mathfrak{A}^I$. We will check that this local time-evolution coincides with the restriction of τ_t^Q on $\mathfrak{C}^I$. To do this we need to find the infinitesimal generator of τ_t^Q. This is a straightforward generalization of the proof in Bóna [**8**].

PROPOSITION 13. *Let E be a mean field connection as above and $Q \in \mathcal{U}(\mathfrak{g}^*)$ an admissible Hamiltonian. The infinitesimal generator δ of τ_t^Q is in the compact case*

$$\delta(E(\hat{f})) = \sum_{i=1}^{n} \int_{supp\ E} E(dF)\Big(\frac{\partial \hat{f}}{\partial F_i}(F)\{Q, F_i\}(F) + \frac{\partial Q}{\partial \beta_i}(F)\delta_{\beta_i}(\hat{f}(F))\Big) \tag{42}$$

where the derivatives of $\hat{f}$ are in the weak sense and δ_{β_i} are the infinitesimal generators of $\tau_t^{X_{\beta_i}}$. In the general case the infinitesimal generator is

$$\begin{aligned} &\omega(\delta(E(\hat{f}))) = \\ &\sum_{i=1}^{n} \int_{Spec\ \mathfrak{N}} \mu_\omega(dm)\rho_\omega(m)\Big(\frac{\partial \hat{f}}{\partial F_i}(F_{\mathfrak{N}}(m))\{Q, F_i\}(F_{\mathfrak{N}}(m)) + \frac{\partial Q}{\partial F_i}(F_{\mathfrak{N}}(m))\delta_{\beta_i}(\hat{f}(F_{\mathfrak{N}}(m)))\Big) \end{aligned} \tag{43}$$

PROOF. We will prove the general case. Again we will use the decomposition (27) of ω. Let us first calculate the derivatives of $\rho_\omega(m)(\hat{f}_t(F_{\mathfrak{N}}(m)))$ for $m \in$ supp μ_ω. For sufficiently regular $\hat{f} \in \mathfrak{C}$ such that $E(\hat{f}) \in D(\delta)$ we have

$$\begin{aligned} &\frac{d}{dt}\Big|_0 \rho_\omega(m)(\sigma(g_Q^{-1}(t, F_{\mathfrak{N}}(m)))\hat{f}(\varphi_t^Q(F_{\mathfrak{N}}(m)))) \\ &\quad = \frac{d}{dt}\Big|_0 \rho_\omega(m)(\hat{f}(\varphi_t^Q(F_{\mathfrak{N}}(m)))) + \frac{d}{dt}\Big|_0 \rho_\omega(m)(\sigma(g_Q^{-1}(t, F_{\mathfrak{N}}(m)))\hat{f}(F_{\mathfrak{N}}(m))). \end{aligned} \tag{44}$$

The first term in (44) is

$$\begin{aligned} &\frac{d}{dt}\Big|_0 \rho_\omega(m)(\hat{f}(\varphi_t^Q(F_{\mathfrak{N}}(m)))) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial F_i}\rho_\omega(m)(\hat{f}(F_{\mathfrak{N}}(m)))\frac{d}{dt}\Big|_0 F_i(\varphi_t^Q(F_{\mathfrak{N}}(m))) = \sum_{i=1}^{n} \frac{\partial}{\partial F_i}\rho_\omega(m)(\hat{f}(F_{\mathfrak{N}}(m)))\{Q, F_i\}(F_{\mathfrak{N}}(m)). \end{aligned}$$

This coincides with the first term in the expression (43). After a similar calculation for the second term in (44) for sufficiently regular $\hat{f} \in \mathfrak{C}$ we have

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \omega(\tau_t^Q E(\hat{f})) \\ &= \sum_{i=1}^{n} \int_{\text{Spec}\ \mathfrak{N}} \mu_\omega(dm)\rho_\omega(m)\Big(\frac{\partial \hat{f}}{\partial F_i}(F_{\mathfrak{N}}(m))\{Q, F_i\}(F_{\mathfrak{N}}(m)) + \frac{\partial Q}{\partial F_i}(F_{\mathfrak{N}}(m))\delta_{\beta_i}(\hat{f}(F_{\mathfrak{N}}(m)))\Big) \end{aligned}$$

which completes the proof. □

THEOREM 4. *The infinitesimal generator δ coincides with the infinitesimal generator δ^I for any finite $I \subset \Pi$.*

PROOF. Let δ be as in (42) and let $x \in \mathfrak{A}^I$ be represented as $\hat{f} \in \mathfrak{C}$. Then since σ acts by inner automorphisms on $\mathfrak{A}^I$ we get that

$$\delta_{\beta_i}(\hat{f}(F)) = i[X_{\beta_i}^I, \hat{f}(F)].$$

The infinitesimal generator (42) is then

$$\delta(E(\hat{f})) = i\sum_{i=1}^{n} \int_{\text{supp}\ E} E(dF)\frac{\partial Q}{\partial \beta_i}(F)[X_{\beta_i}^I, \hat{f}(F)].$$

This is exactly the generator of $\tau_t^{Q^I}$. The non-compact case is similar. □

6. Concluding remarks

Let us consider the setup of Example 1. Since G is compact and U is norm-continuous the generators X_β are bounded for any $\beta \in \mathfrak{g}$. The Cesaro means of the generators X_{β_i} for any $I \subset \Pi$ finite are then bounded and the same is true of the local Hamiltonians Q^I given by (36). It was proven in Bóna [**8**] that the limit

$$\tilde{\tau}_t^Q := s^* - \lim_I \tau_t^{Q^I}$$

exists by using uniform (in $I \subset \Pi$) norm convergence of the series expansion of the local time-evolutions $\tau_t^{Q^I}$

$$\tau_t^{Q^I}(x) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \underbrace{[Q^I, \dots, [Q^I, x] \dots]}_{m \text{ times}} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \mathrm{ad}_{Q^I}^m x \tag{45}$$

for sufficiently small time. The norm-bounds of the iterated commutator $\mathrm{ad}_{Q^I}^m x$ are obtained writing it as a polynomial in x and the generators $X_{\beta_i I}$ using the Lie algebra commutation relations (18). Since $X_{\beta\Pi}$ are limits $\mathrm{s} - \lim_I X_{\beta I} p_G$ one has that the s^*-limits of the iterated commutators $\mathrm{ad}_{Q^I}^m x$ exist in $p_G \mathfrak{A}^{**}$. Then one can then define the s^*-limit of the $\tau_t^I(x)$ by (45).

An important result of Bóna [**8**] is then that τ_t^Q coincides with $\tilde{\tau}_t^Q$. We conjecture this is also true in the non-compact case. Since Q^I are unbounded in the general case the method of expansion (45) does not apply. Instead one can possibly use restrictions of Q^I or investigate the action of one-dimensional Lie algebra acting by $[Q^I, \cdot]$ on $p_G \mathfrak{A}^{**}$ by unbounded operators.

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