Diploma Thesis

Gravitational Characteristics of Galaxies
Comenius University
Faculty of Mathematics, Physics and Informatics

Gravitational Characteristics of Galaxies
Diploma thesis

Astronomy and Astrophysics

4.1.1. Physics

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Abstract

[diploma thesis]
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The problematics of the galactic halo is discussed in this diploma thesis. Two kinds of models are used. The first kind describes galactic halo of the Galaxy and it is in accordance with the observational data (e.g. the Oort constants, flat rotation curve). The second kind is based on the observations of other galaxies. A new model of the galactic halo, which combines both kinds of models, is created. The new model gives observed values of the Oort constants in the solar region, it produces the flat rotation curve and satisfies the conditions based on the observations of other galaxies. Properties of the models are presented, too. The solar motion in our Galaxy is discussed in the last chapter and some applications of the solar motion are pointed out.

Keywords: Galaxy, mass density, dark matter
Abstrakt

NAGY Roman. Gravitačné charakteristiky galaxií.

[diplomová práca]
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Kľúčové slová: Galaxia, hustota hmotnosti, tmavá hmota
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Introduction

The universe is the only one where we observe the effects of all known physical laws in one place, from the quantum theory to the general theory of relativity. One of the phenomena which connects these limiting theories is the dark matter.

The dark matter problem is one of the biggest open questions in physics. Most of the cosmological theories request its existence, but none of them explains its nature. This is the main reason why we want to deal with the problematics of the dark matter. As we do not know its structure and composition, we have to study observable effects of the dark matter. One of them is a flat rotation curve of galaxies. This means that a circular speed of galactic objects do not depend on galactocentric distances, if the objects move in circular orbits situated in the galactic equatorial plane.

Our goal is to create a realistic model of galaxies, which can properly describe gravitational effects of a visible matter and the dark matter, too. We will focus on galactic halos, where the gravitational attraction of the dark matter is strongest.

Our model of the galaxy has to be in accordance with the observational data of our Galaxy and other galaxies, too. Thus we will combine the realistic model of our Galaxy with another model, which properly describes the other galaxies. The final model should provide realistic approach to the gravitation of each galaxy, respecting its dark matter.
Chapter 1

Galactic models

Our prime goal is to create a new realistic model of Galaxy. Our model has to be consistent with the observational data.

At first, our new realistic model has to produce a flat rotation curve. This means that a circular speed of objects does not depend on galactocentric distances, if the objects move in circular orbits situated in the galactic equatorial plane. Moreover, the model has to yield the measured values of the Oort constants in the region of the Sun.

Galaxy consists of a galactic bulge, disk and halo. Each part of the Galaxy is represented by our own model. We will focus on the galactic halo, because the effect of dark matter is strong in this part of the Galaxy.

Now we present some models which will be used in this thesis.

1.1 Dauphole et al. (1996)

Dauphole et al. (1996) present the model of the galactic bulge, disk and halo. The potential model consists of a gravitational potential of the galactic bulge \( \Phi_b \), disk \( \Phi_d \) and the halo \( \Phi_h \):

\[
\Phi_b = -\frac{\kappa M_b}{\sqrt{r^2 + b_b^2}},
\]

\[
\Phi_h = -\frac{\kappa M_h}{\sqrt{r^2 + b_h^2}},
\]
\[ \Phi_d = -\frac{\kappa M_d}{\sqrt{R^2 + (a_d + \sqrt{z^2 + b_d^2})^2}}, \]  

(1.3)

where the values of the constants are:

\[ M_b = 1.3955 \times 10^{10} \, M_\odot, \]
\[ b_b = 0.35 \, \text{kpc}, \]
\[ M_h = 6.9776 \times 10^{11} \, M_\odot, \]
\[ b_h = 24 \, \text{kpc}, \]
\[ M_d = 4.9080 \times 10^{10} \, M_\odot, \]
\[ b_d = 0.25 \, \text{kpc}, \]
\[ a_d = 3.55 \, \text{kpc}, \]

(1.4)

and \( \kappa \) is the gravitational constant. The circular speed of the potential model is presented by Nagy (2009)

\[ v(r) = \sqrt{\frac{\kappa R^2}{\left[ R^2 + (a_d + b_d)^2 \right]^{3/2}} + \frac{M_b}{\left[ R^2 + b_h^2 \right]^{3/2}} + \frac{M_d}{\left[ R^2 + b_b^2 \right]^{3/2}}}. \]  

(1.5)

The rotation curve of the potential model is presented in Fig. 1.1.

Figure 1.1: Rotation curve of the potential model for \( R \in < 0, 200 > \, \text{kpc}. \)
As we can see in Fig. 1.1, the rotation curve of the potential model is a decreasing function for galactocentric distances greater than about 40 kpc.

Oort constants calculated from the potential model are presented in Nagy (2009). The values of the Oort constants in the solar region (approximately 8 kpc from the galactic center) calculated from the model are:

\[ A = 14.2486 \text{ km s}^{-1} \text{ kpc}^{-1}, \quad B = -13.8876 \text{ km s}^{-1} \text{ kpc}^{-1}. \] (1.6)

Klačka (2009) presents the most probable values of A and B

\[ A = (14.2 \pm 0.5) \text{ km s}^{-1} \text{ kpc}^{-1}, \quad B = (-12.4 \pm 0.5) \text{ km s}^{-1} \text{ kpc}^{-1}. \] (1.7)

These values were calculated by using Oort constants taken from various sources. The values (1.6) are not fully consistent with values (1.7).

Finally, the potential model of the Galaxy does not lead to the flat rotation curve and the Oort constants are not fully in accord with (1.7). Thus, the potential model is not enough realistic to be used.

1.2 Klačka (2009)

Klačka (2009) presents the model for each part of the Galaxy, too. He uses the potential model for the bulge of the Galaxy given by Eq. (1.1) which is in accordance with a conception of the spherical symmetric bulge of the Galaxy. The circular speed of this model is presented by (Nagy 2009, pp. 25)

\[ v_{\text{bulge}} = \sqrt{\frac{nR^2M_b}{[R^2 + b^2_{\text{d}}]^{3/2}}}, \] (1.8)

the constants are presented in (1.4).

For the galactic disk he uses the model presented by Maoz (2007). This model is based on the distribution of the mass density \( \rho_{\text{disk}}(R, z) \) in the galactic disk

\[ \rho_{\text{disk}}(R, z) = \rho_0 \exp \left( -\frac{R}{R_d} \right) \exp \left( -\frac{|z|}{h_d} \right), \] (1.9)

where \( R, z \) are cylindrical coordinates, \( \rho_0 \) is a central mass density, \( R_d = (3.5 \pm 0.5) \text{ kpc} \) is the scale-length of the disk and \( h_d \) is the characteristic height which reaches two values. For
the stars in the disk is \( h_d = 330 \text{ pc} \) and for the dust and gas is \( h_d = 160 \text{ pc} \). This model assumes a cylindrical symmetry of the galactic disk. The circular speed is presented by Binney and Tremaine (1987, pp. 78) and for this model it is calculated in Nagy (2009, pp. 32):

\[
v_{\text{disk}} = \left\{ 4\pi \kappa \Sigma_0 \frac{R^2}{4R_d} \left[ I_0 \left( \frac{R}{2R_d} \right) K_0 \left( \frac{R}{2R_d} \right) - I_1 \left( \frac{R}{2R_d} \right) K_1 \left( \frac{R}{2R_d} \right) \right] \right\}^{1/2},
\]

(1.10)

\[\Sigma_0 = \frac{0.97 \times 10^{10}}{2\pi (1 - e^{-1} + 2e^{-2}) R_d^2},\]

where \( I_n \) are the modified Bessel functions of the first kind and \( K_n \) are the modified Bessel functions of the second kind.

A new model for the galactic halo is defined. It is given by the circular speed \( v_{\text{circ}}(r) \)

\[
v_{\text{halo}}^2 = v_H^2 \{ 1 - \alpha a_H \frac{r}{r_a} \arctan \left( \frac{r}{a_H} \right) - (1 - \alpha) \exp \left( \frac{-r^2}{b_H^2} \right) \},
\]

(1.11)

the constants are presented below Eq. (2.1). The calculated numerical values of \( a_H \) and \( b_H \) cause that the Oort constants \( A, B \) will be consistent with values (1.7). \( \alpha \) is a free parameter which was calculated on the basis of the principle of minimum potential energy. Thus we can calculate the \( \alpha \) parameter from various physical conditions.

![Figure 1.2: Rotation curve of the model of the Galaxy presented by Klačka (2009) for \( R \in (0, 100) \text{ kpc} \).](image-url)
CHAPTER 1. GALACTIC MODELS

The circular speed of the Galaxy is:

\[ v(R) = \sqrt{v_{\text{bulge}}^2(R) + v_{\text{disk}}^2(R) + v_{\text{halo}}^2(R)} . \]  
(1.12)

By using Eq. (1.8)-(1.12) we get:

\[ v(R) = \left\{ \kappa R^2 M_b \left[ R^2 + b_0^2 \right]^{-3/2} + v_H^2 \left( 1 - \alpha \frac{2H}{r} \right) \arctan \left( \frac{r}{\alpha H} \right) - (1 - \alpha) \exp \left( -\frac{r^2}{b^2} \right) \right\} + \pi \kappa \Sigma_0 R^2 R_d^{-1} \left[ I_0 \left( \frac{R}{2R_d} \right) K_0 \left( \frac{R}{2R_d} \right) - I_1 \left( \frac{R}{2R_d} \right) K_1 \left( \frac{R}{2R_d} \right) \right]^{1/2} . \]  
(1.13)

The rotation curve of the Galaxy is presented in Fig. 1.2.

As we can see in Fig. 1.2 this model produces the flat rotation curve. Moreover, the Oort constants A, B are consistent with values (1.7). Thus this model is a good basis for the realistic model of the Galaxy.

1.3 Burkert (1995)

Burkert (1995) presents the volume mass density of the galactic halo

\[ \rho(r) = \frac{\rho_0 r_0^3}{(r + r_0)(r^2 + r_0^2)} , \]  
(1.14)

where \( r_0 \) is a scale radius and \( \rho_0 \) is the central volume mass density. The quantities \( r_0 \) and \( \rho_0 \) are free parameters which fit the observational data. The circular speed of this model is:

\[ v(r) = \sqrt{\kappa \pi \rho_0 r_0^2 \left\{ \frac{r_0}{r} \ln \left( \frac{r_0^2 + r^2}{r_0^2} \right) \left( r_0 + r \right)^2 - 2 \frac{r_0}{r} \arctan \left( \frac{r}{r_0} \right) \right\}} . \]  
(1.15)

Salucci et al. (2007) calculated the quantities \( r_0 \) and \( \rho_0 \) from their set of data and the rotation curve of Burkert model is presented in Figure 3 in Salucci et al. (2007). We see that the rotation curves are not flat, therefore this model does not completely describe the galactic halo.

Models presented by Donato et al. (2009) and Gentile et al. (2009), which are based on Burkert model of the galactic halo, are discussed in the following chapters. Donato et al. (2009) and Gentile et al. (2009) exposed that Burkert model properly describes another physical quantities of the galactic halos, therefore we use these articles in this thesis, too.
1.4 Some other models

In this section we mention some models of the galactic halo which are frequently used. However, they will be not used in this thesis. The reason is that they do not fit data of Galaxy.

1.4.1 Begeman et al. (1991)

Begeman et al. (1991) present the mass volume density of the halo

\[ \rho(r) = \frac{\rho_0}{1 + \left( \frac{r}{r_0} \right)^2}, \]  

(1.16)

where \( r_0 \) and \( \rho_0 \) are free parameters. The circular speed is

\[ v(r) = \sqrt{4\pi \kappa \rho_0 r_0^2 \left( 1 - \frac{r_0}{r} \arctan \left( \frac{r}{r_0} \right) \right)}. \]  

(1.17)

1.4.2 Navarro et al. (1996)

Navarro et al. (1996) present the volume mass density of the dark matter halo

\[ \rho(r) = \frac{\rho_c}{\frac{r}{r_0} + \left( \frac{r}{r_0} \right)^2}, \]  

(1.18)

where \( r_0 \) and \( \rho_c \) are free parameters. The circular speed is

\[ v(r) = \sqrt{4\pi \kappa \rho_0^2 \rho_c \left( \frac{r_0}{r} \ln \left( 1 + \frac{r}{r_0} \right) \right)}. \]  

(1.19)
Chapter 2

Model of galactic halo

We will consider spherically symmetric mass distribution in a galactic halo, as it is conventionally done. We want to improve a model of the galactic halo by using recent discoveries about universality of the dark matter halo surface density in galaxies (Gentile et al. 2009). We will use the model of the galactic halo presented by Klačka (2009).

2.1 Rotation curve of the galactic halo

The model presented by Klačka (2009) defines a circular speed $v_{\text{circ}}(r)$ for halo of the Galaxy:

$$v_{\text{circ}}^2 = v_H^2 \left( 1 - \alpha \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) - (1 - \alpha) \exp \left( -\frac{r^2}{b^2} \right) \right),$$

where

- $r$ is the distance from the galactic center,
- $v_H = 220 \text{ km.s}^{-1}$,
- $a_H = 0.04383 \text{ kpc}$,
- $b = b_H = 37.3760 \text{ kpc}$,
- $\alpha = 0.174$.

The numerical value of a free parameter $\alpha$ was calculated on the basis of the principle of minimum potential energy.
In our diploma thesis we want to calculate the value of the $\alpha$ parameter from various conditions. We will concentrate on observational data.

### 2.2 $r_0\rho_0$

Donato et al. (2009) present that the product $r_0\rho_0$ is the same for all galaxies. This assumption is shown in Eq. (2.2):

$$r_0\rho_0 = 141 \pm 52 \ M_\odot pc^{-2}, \tag{2.2}$$

where the central mass volume density is $\rho_0$ and the distance $r_0$ is defined by the condition:

$$\rho(r_0) = \frac{\rho_0}{4}. \tag{2.3}$$

### 2.3 Surface density

Gentile et al. (2009) present the universality of galactic surface densities. Gentile et al. (2009) claim that the mean dark matter surface density $\langle \Sigma_{0,dark} \rangle$ within a scale-length is the following value for various galaxies:

$$\langle \Sigma_{0,dark} \rangle = \frac{M_{<r_0}}{\pi r_0^2} = 72 \pm 42 \ M_\odot pc^{-2}, \tag{2.4}$$

where $r_0$ is the scale-length, i.e. radius at which the local dark matter volume density reaches a quarter of its central value (see Eq. (2.3)). $M_{<r_0}$ is mass of the matter within the one scale-length radius $r_0$.

However, the results (2.2) and (2.4) hold only for a mass modeling using a Burkert (1995) dark matter halo:

$$\rho_{DM}(r) = \frac{\rho_0 r_0^3}{(r + r_0)(r^2 + r_0^2)}, \tag{2.5}$$

where the quantities $\rho_0$ and $r_0$ are free parameters which represent the central density and the scale radius, respectively.
Chapter 3

Gravitational potential approach

We will use Poisson’s equation for gravitational potential:

\[ \Delta \Phi = 4\pi \kappa \rho , \tag{3.1} \]

where \( \Phi \) is a gravitational potential, \( \rho \) is a mass density and \( \kappa \) is gravitational constant. Writing Eq. (3.1) in cylindrical coordinates we get:

\[ \Delta \Phi = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi \kappa \rho , \tag{3.2} \]

where \( R, \theta \) and \( z \) are cylindrical coordinates. Then we define surface density \( \Sigma(R) \) as a function of \( R \):

\[ \Sigma(R) = \int_{-\infty}^{\infty} \rho(R, z) \, dz , \tag{3.3} \]

where \( \rho(R, z) \) is volume mass density. By using Eq. (3.2) in Eq. (3.3) we get:

\[ \Sigma(R) = \int_{-\infty}^{\infty} \left[ \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] \, dz . \tag{3.4} \]

We assume, that the gravitational potential of galactic halo is an independent function of cylindrical coordinate \( \theta \), thus

\[ \frac{\partial \Phi}{\partial \theta} = 0 \tag{3.5} \]

Then, rewriting Eq. (3.4) by using Eq. (3.5), we get:

\[ \Sigma(R) = \frac{1}{4\pi \kappa} \left\{ \int_{-\infty}^{\infty} \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) \, dz + \left[ \frac{\partial \Phi}{\partial z} \right]_{-\infty}^{\infty} \right\} . \tag{3.6} \]
The gravitational potential of the galactic halo is symmetric with respect to the galactic equator. This means that

$$\left[ \frac{\partial \Phi}{\partial z} \right]_{-\infty}^{\infty} = 0.$$  \hfill (3.7)

Moreover, each of the values fulfills \( \lim_{x \to \infty} (\partial \Phi / \partial z) = \lim_{z \to -\infty} (\partial \Phi / \partial z) = 0 \). Hence, we can rewrite Eq. (3.6) into

$$\Sigma(R) = \frac{1}{4\pi \kappa} \int_{-\infty}^{\infty} 1 R \frac{\partial}{\partial R} (R \frac{\partial \Phi}{\partial R}) \, dz .$$  \hfill (3.8)

The mean surface density within \( r_0 \) we define as

$$\langle \Sigma \rangle_0 = \frac{1}{r_0^2} \int_0^{r_0} \Sigma(R) \, dS = \frac{1}{r_0^2} \int_0^{r_0} 2\pi R \Sigma(R) \, dR .$$  \hfill (3.9)

From Eq. (3.8) and Eq. (3.9) we get

$$\langle \Sigma \rangle_0 = \frac{2}{4\pi \kappa r_0^2} \int_0^{r_0} R \left[ \int_{-\infty}^{\infty} 1 R \frac{\partial}{\partial R} (R \frac{\partial \Phi}{\partial R}) \, dz \right] \, dR .$$  \hfill (3.10)

We also know that circular speed is equal to

$$v_{\text{circ}} = \left( R \frac{\partial \Phi}{\partial R} \right)^{1/2} , \quad z = 0 .$$  \hfill (3.11)

We will proceed in the following way. At first,

$$R \frac{\partial \Phi}{\partial R} = R \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial R} = \left( \frac{R}{r} \right)^2 r \frac{\partial \Phi}{\partial r} , \quad r = \sqrt{r_0^2 + z^2} .$$  \hfill (3.12)

For spherical potential we can write, in analogy to Eq. (3.11),

$$v_{\text{circ}} = \left( r \frac{\partial \Phi}{\partial r} \right)^{1/2} .$$  \hfill (3.13)

This holds not only for \( z = 0 \). By using Eqs. (3.12) and (3.13) we get:

$$R \frac{\partial \Phi}{\partial R} = \frac{R^2}{R^2 + z^2} \frac{r^2}{v_{\text{circ}}} .$$  \hfill (3.14)

Thus, Eq. (3.10) leads to:

$$\langle \Sigma \rangle_0 = \frac{1}{2\pi \kappa r_0^2} \int_{-\infty}^{\infty} \frac{r_0^2}{r_0^2 + z^2} \left[ v_{\text{circ}} \left( r = \sqrt{r_0^2 + z^2} \right) \right]^2 \, dz .$$  \hfill (3.15)
3.1 Volume mass density approach

For a circular motion and a spherically symmetric mass distribution, we can write:

\[
\frac{v^2}{r} = \frac{4\pi \kappa \int_0^r r^2 \rho(r) \, dr}{r^2}.
\]  

(3.16)

This yields:

\[
\rho(r) = \frac{1}{4\pi \kappa} \frac{1}{r^2} \frac{d}{dr} \{r [v(r)]^2\},
\]  

(3.17)

where \( r = \sqrt{R^2 + z^2} \) in cylindrical coordinates. By using Eq. (3.3) we get:

\[
\Sigma(R) = \frac{1}{4\pi \kappa} \int_{-\infty}^{\infty} \frac{1}{R^2 + z^2} \frac{d}{dr} \{r [v(r)]^2\} \, dz, \quad r = \sqrt{R^2 + z^2},
\]  

(3.18)

Again, as in Eq. (3.9), we can write for the mean surface density:

\[
\langle \Sigma \rangle_0 = \frac{1}{\pi} \int_0^{r_0} 2\pi R \Sigma(R) \, dR.
\]  

(3.19)

Hence, Eqs. (3.18) and (3.19) yield:

\[
\langle \Sigma \rangle_0 = \frac{1}{r_0^2 2\pi \kappa} \int_0^{r_0} R \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2 + z^2} \frac{d}{dr} \{r [v(r)]^2\} \, dz \right\} \, dR, \quad r = \sqrt{R^2 + z^2}.
\]  

(3.20)

This result leads to Eq. (3.15) again. However, numerical calculations have to be done to prove it. Thus everything is all right.

The relation between \( \alpha \) and the mean surface density \( \langle \Sigma \rangle_0 \) is presented in Fig. 3.1.

Figure 3.1: The relation between the mean surface density and \( \alpha \)
As a special case we introduce

$$\lim_{r_0 \to 0} \langle \Sigma \rangle_0 = \frac{v_H^2}{2\pi \kappa} \left[ \frac{2\pi (1 - \alpha)}{b} + \frac{\alpha \pi}{2a} \right].$$

(3.21)

We remind that $\langle \Sigma \rangle_0$ is not equivalent to $\langle \Sigma_{0,\text{dark}} \rangle$ given by (2.4). However, the surface density calculated in this chapter is based on the conventionally used definition of the surface density.
Chapter 4

Simple model of the galactic halo

(\alpha \equiv 1)

This chapter deals with the conventional access used for mass volume density of a galactic halo.

4.1 Mass volume density

Galactic halo may be described with the mass volume density:

\[ \rho_H = \frac{v_H^2}{4\pi\kappa} \left( r^2 + a_H^2 \right)^{-1}. \]  \hspace{1cm} (4.1)

"This density law is often used to represent the mass of galaxy’s dark halo.” (Sparke and Gallagher 2007, p.95). Numerical value of the constant \( v_H \) is presented below Eq. (2.1).

4.2 Rotation curve of galactic halo

Now we can use Poisson’s equation for gravitational potential in spherical coordinates for the case of spherical symmetry:

\[ 4\pi\kappa\rho = \triangle \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right). \]  \hspace{1cm} (4.2)
We can use the knowledge of spherical symmetry of the galactic halo. The condition for centripetal acceleration yields:

\[ v_{\text{circ}} = \left( r \frac{\partial \Phi}{\partial r} \right)^{1/2}, \]  

which is similar to Eq.(3.16). By using Eqs.(4.2) and (4.3) we get:

\[ v_{\text{circ}}^2(r) = \frac{4\pi \kappa}{r} \int_0^r x^2 \rho(x) \, dx, \]  

Then, rewriting Eq.(4.4) by using Eq.(4.1), we get:

\[ v_{\text{circ}}^2(r) = \frac{v_H^2}{r} \int_0^r \frac{x^2}{x^2 + a_H^2} \, dx. \]  

Eq. (4.5) leads to:

\[ v_{\text{circ}}^2(r) = v_H^2 \left[ 1 - \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) \right]. \]  

This circular speed is a special case of Eq. (2.1) for \( \alpha = 1 \). The rotation curve is presented in Fig. 4.1 and Fig. 4.2.
4.3 Surface density

We can calculate the surface density by using Eqs. (3.3) and (4.1):

\[ \Sigma(R) = \frac{v_H^2}{4\kappa a_H} \left( 1 + \frac{R^2}{a_H^2} \right)^{-1/2}. \]  \hspace{1cm} (4.7)

From Eqs. (3.19) and (4.7) we get the mean surface density:

\[ \langle \Sigma \rangle_0 = \frac{v_H^2 a_H}{2 \kappa r_0^2} \left( \sqrt{1 + \frac{r_0^2}{a_H^2}} - 1 \right), \]  \hspace{1cm} (4.8)

where \( r_0 \) is a scale-length. The scale-length is defined (Donato et al. 2009) as a radius, at which the local dark matter volume density reaches a quarter of its central value. Eq. (4.1) yields:

\[ r_0 = \sqrt{\frac{3}{a_H}}. \]  \hspace{1cm} (4.9)

Now we can calculate the mean surface density on the basis of Eqs. (4.8)-(4.9):

\[ \langle \Sigma \rangle_0 = \frac{v_H^2}{6\kappa a_H} = \frac{\pi}{3\sqrt{3}} \rho_0 r_0, \]  \hspace{1cm} (4.10)
or, numerically:

$$\langle \Sigma \rangle_0 = 4.28 \times 10^4 \ M_\odot \ pc^{-2} \ .$$  \hspace{1cm} (4.11)

As we can see in Eq. 1.10, the mean surface density \( \langle \Sigma \rangle_0 \) depends linearly on the product \( \rho_0 r_0 \). Thus the standard deviation of the mean surface density \( \sigma_{\langle \Sigma \rangle_0} \) depends on the standard deviation of that product \( \sigma_{\rho_0 r_0} \) linearly too. Then:

$$\sigma_{\langle \Sigma \rangle_0} = \frac{\pi}{3\sqrt{3}} \sigma_{\rho_0 r_0} \ .$$ \hspace{1cm} (4.12)

4.4 Donato et al. (2009)

Donato et al. (2009) present that the product \( r_0 \rho_0 \) is the same for all galaxies. This statement was shown in the section 2.2. Thus:

$$r_0 \rho_0 = 141 \pm 82 \ M_\odot pc^{-2} ,$$ \hspace{1cm} (4.13)

where the scale-length \( r_0 \) and the central mass volume density \( \rho_0 \) are defined by Eq. 2.3.

4.4.1 Finding \( a_H \)

We calculate the alternative value of the parameter \( a_H \). By using Eqs. 2.3, 4.1, 4.9 and 4.13 we get:

$$\rho_0 r_0 = \frac{v_H^2}{4\pi \kappa a_H^2} \cdot \sqrt{3} a_H ,$$ \hspace{1cm} (4.14)

where \( v_H = 220 \ km.s^{-1} \). Thus:

$$a_H = \sqrt{\frac{3}{4\pi}} \frac{v_H^2}{\kappa \rho_0 r_0} ,$$ \hspace{1cm} (4.15)

where the product \( \rho_0 r_0 \) is given by Eq. 4.14. The standard deviation of \( a_H \) is:

$$\sigma_{a_H} = a_H \frac{\sigma_{\rho_0 r_0}}{\rho_0 r_0} ,$$ \hspace{1cm} (4.16)

where \( \sigma_{\rho_0 r_0} \) is the standard deviation of the product \( \rho_0 r_0 \). \( \sigma_{\rho_0 r_0} \) is estimated from 4.13 as:

$$\sigma_{\rho_0 r_0} = 67 \ M_\odot pc^{-2} \ .$$ \hspace{1cm} (4.17)

Finally, Eqs. 4.12, 4.15–4.17 lead to

$$a_H = 11 \pm 5.227 \ \text{kpc} \ .$$ \hspace{1cm} (4.18)
4.4.2 Rotation curve

The circular speed is given by Eq. (4.6) and the rotation curve of the simple model with $a_H$ given by (4.18) is presented in Fig. 4.3.

Figure 4.3: Rotation curve of the simple for $a_H$ calculated from the $\rho_0 r_0$ product. $a_H = 11$ kpc - solid line, for $a_H = 5.773$ kpc - dotted line, for $a_H = 16.227$ kpc - dashed line. $R \in <0, 100>$ kpc
Chapter 5

Improved model of the galactic halo

\( (\alpha = 0.174) \)

This chapter deals with the mean surface density of the model represented by circular speed (2.1).

5.1 Mass volume density

From Eqs. (2.1) and (3.17) we get the volume mass density:

\[
\rho(r) = \frac{v_H^2}{4\pi\kappa} \left[ \frac{1}{r^2} - \frac{\alpha}{1 + \frac{a_H^2}{a_H^2}} \frac{1}{r^2} + \frac{(\alpha - 1)e^{-\frac{r^2}{b^2}}}{r^2} \left( 1 - \frac{2r^2}{b^2} \right) \right].
\]

(5.1)

Also:

\[
\lim_{r \to 0} \rho = \frac{v_H^2}{4\pi\kappa} \left( \frac{3 - 3\alpha}{b^2} + \frac{\alpha}{a_H^2} \right) = \rho_0,
\]

(5.2)

which represents the central volume mass density.

5.2 Rotation curve of galactic halo

The improved model \((\alpha = 0.174)\) is represented by circular speed:

\[
v_{circ}^2 = v_H^2 \{ 1 - \alpha \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) - (1 - \alpha) \exp \left( -\frac{r^2}{b^2} \right) \}.
\]

(5.3)
Thus the rotation curve of this model is presented in Fig. 5.1 and Fig. 5.2.

Figure 5.1: Rotation curve of the improved model for $R \in (0, 10) \text{ kpc}$

Figure 5.2: Rotation curve of the improved model for $R \in (10, 100) \text{ kpc}$
CHAPTER 5. IMPROVED MODEL OF THE GALACTIC HALO ($\alpha = 0.174$)

5.3 Surface density

By using Eqs. (3.18) and (2.1) we obtain

$$\Sigma(R) = \frac{v_H^2}{4\pi \kappa} \left[ \frac{\pi \alpha}{\sqrt{R^2 + a_H^2}} + (1 - \alpha) \frac{2\sqrt{\pi} e^{-\frac{R^2}{b^2}}}{b} - (\alpha - 1) \frac{\pi}{R} \text{Erf} \left( \frac{R}{b} \right) \right] ,$$

(5.4)

where

$$\text{Erf} (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt .$$

(5.5)

Then, from Eqs. (3.19) and (5.4) we get the mean surface density:

$$\langle \Sigma \rangle_0 = \frac{v_H^2}{2\kappa r_0^2} \left[ \alpha \left( \sqrt{a_H^2 + r_0^2} - a_H \right) + (1 - \alpha)r_0 \text{Erf} \left( \frac{r_0}{b} \right) \right] ,$$

(5.6)

The scale-length for this model is:

$$r_0 = 75.92 \text{ pc} ,$$

(5.7)

since:

$$\rho(r_0) = \frac{\rho_0}{4}$$

(5.8)

on the basis of Eqs (5.1) and (5.2). The mean surface density:

$$\langle \Sigma \rangle_0 = 7.45 \times 10^3 \ M_\odot \text{ pc}^{-2} .$$

(5.9)
Chapter 6

Donato et al. (2009) - $\rho_0 r_0$

Donato et al. (2009) claim that the product $r_0 \rho_0$ is the same for all galaxies. This statement was shown in the section 2.2. Thus:

$$r_0 \rho_0 = 141 \pm 62 \ M_\odot pc^{-2}, \quad (6.1)$$

where the scale-length $r_0$ and the central mass volume density $\rho_0$ are defined by:

$$\rho(r_0) = \frac{\rho_0}{4}. \quad (6.2)$$

6.1 Simple model ($\alpha \equiv 1$)

In this case we use the mass volume density and the scale-length which are given by Eqs. (4.1) and (4.9). Hence

$$\rho_0 r_0 = \frac{v_H^2}{4\pi \kappa a_H^2} \sqrt{3} \ a_H, \quad (6.3)$$

where the constants are presented below Eq. (2.1). The numerical value of the product is

$$r_0 \rho_0 = 3.54 \times 10^4 \ M_\odot pc^{-2}. \quad (6.4)$$

6.2 Improved model ($\alpha = 0.174$)

The volume mass density is given by Eq. (5.1). The central mass volume density has been calculated in Chapter 5 and it is shown in Eq. (5.2). The scale-length is presented in Eq. (5.7).
Thus, we can write

\[ r_0 \rho_0 = r_0 \left( \frac{2.478}{b^2} + \frac{0.174}{a_H^2} \right) \frac{v_H^2}{4\pi \kappa}, \]  

which leads to

\[ r_0 \rho_0 = 6.16 \times 10^3 \, M_\odot pc^{-2}. \]  

6.3 Finding new \( \alpha \)

In this subsection we want to calculate the numerical value of the \( \alpha \) parameter from the condition represented by Eq.\( (6.1) \), thus we create the model, which combines the models presented by Klačka (2009) and Donato et al. (2009). We use the mass volume density defined in Eq.\( (5.1) \).

The central mass volume density is in Eq.\( (5.2) \) and the scale-length is determined by Eqs.\( (5.1)-(5.2) \) and \( (7.14) \) as:

\[ \frac{v_H^2}{4\pi \kappa} \left[ \frac{1}{r_0^2} - \frac{\alpha}{1 + \frac{r_0^2}{a_H^2}} \frac{1}{r_0^2} + \frac{(\alpha - 1)e^{-\frac{r_0^2}{b^2}}}{r_0^2} \left( 1 - \frac{2r_0^2}{b^2} \right) \right] = \frac{1}{4} \frac{v_H^2}{4\pi \kappa} \left( \frac{3 - 3\alpha}{b^2} + \frac{\alpha}{a_H^2} \right). \]  

Now we want to satisfy the condition \( (6.1) \) and after numerical calculations we get the \( \alpha \) parameter of the new model of the galactic halo as:

\[ \alpha = 0.00397^{+0.00232}_{-0.00147}. \]  

The errors in Eq. \( (6.8) \) are estimated by using extremal values of \( \rho_0 r_0 \) (\( \rho_0 r_0 = 141 + 82 \, M_\odot pc^{-2} \) and \( \rho_0 r_0 = 141 - 52 \, M_\odot pc^{-2} \)) in the calculations.

The rotation curve of this model is presented in Fig. 6.1 and Fig. 6.2. The relation between \( \alpha \) and the product \( \rho_0 r_0 \) is presented in Fig. 6.3.
Figure 6.1: Rotation curve of the new model \((r_0\rho_0)\) for \(\alpha = 0.00397\) - solid line, for \(\alpha = 0.0025\) - dotted line, for \(\alpha = 0.0063\) - dashed line.
\(R \in <0,10> \text{ kpc}\)

Figure 6.2: Rotation curve of the new model \((r_0\rho_0)\) for \(\alpha = 0.00397\) - solid line, for \(\alpha = 0.0025\) - dotted line, for \(\alpha = 0.0063\) - dashed line.
\(R \in <10,100> \text{ kpc}\)
Figure 6.3: The relation between $\alpha$ and $\rho_0 r_0$. 

$\rho_0 r_0 \ [M_{\text{sun}} \ pc^{-2}]$

$\alpha$

$0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0$
Gentile et al. (2009) claim that the mean surface density within a scale-length $r_0$ is a constant independent on individual galaxies. This assumption is shown in Eq. (2.4). But the authors defined the surface density in the other way. Their definition of the surface density is:

$$\langle \Sigma_{0,dark} \rangle = \frac{M_{<r_0}}{\pi r_0^2} = 72 \pm 42 \, M_\odot pc^{-2}, \quad (7.1)$$

where $r_0$ is the scale-length and $M_{<r_0}$ is mass of matter within a sphere whose radius is equal to one scale-length. Thus we calculated the surface density in this way, too.

We have

$$\langle \Sigma_{0,dark} \rangle = \frac{1}{\pi r_0^2} \int_{V_0} \rho(r) dV , \quad (7.2)$$

where $V_0$ is the volume of the sphere whose radius is equal to one scale-length $r_0$. We can write for spherical symmetry:

$$\langle \Sigma_{0,dark} \rangle = \frac{1}{\pi r_0^2} \int_0^{r_0} 4\pi r^2 \rho(r) dr . \quad (7.3)$$

### 7.1 Simple model ($\alpha \equiv 1$)

We calculate Gentile’s surface density for simple model defined by the value of the parameter $\alpha$, $\alpha = 1$. By using Eqs. (4.1) and (7.3) we get:

$$\langle \Sigma_{0,dark} \rangle = \frac{v_H^2}{\pi \kappa r_0^2} \int_0^{r_0} \frac{r^2}{r^2 + a_H^2} \, dr . \quad (7.4)$$
After integration we get:

$$\langle \Sigma_{0, \text{dark}} \rangle = \frac{v_H^2}{\pi \kappa} \left[ \frac{1}{r_0} - \frac{a_H}{v_H^2} \arctan \left( \frac{r_0}{a_H} \right) \right].$$

(7.5)

Numerical values of the constants are presented in Eq. (4.9) and below Eq. (2.1). Hence,

$$\langle \Sigma_{0, \text{dark}} \rangle = 1.86 \times 10^4 \ M_\odot \ pc^{-2}.$$  

(7.6)

### 7.2 Finding $a_H$

By using Eqs. (7.1) and (7.5) we numerically calculate the alternative value of the $a_H$ parameter.

The standard deviation of $a_H \sigma_{a_H}$ is defined by Eq. (7.7):

$$\sigma_{\langle \Sigma_{0, \text{dark}} \rangle} = \left| \frac{\partial \langle \Sigma_{0, \text{dark}} \rangle}{\partial a_H} \right| \sigma_{a_H} = \frac{v_H^2}{\pi \kappa a_H^2} \left( \frac{1}{\sqrt{3}} - \frac{\arctan \sqrt{3}}{3} \right) \sigma_{a_H},$$

(7.7)

where $\sigma_{\langle \Sigma_{0, \text{dark}} \rangle}$ is the standard deviation of $\langle \Sigma_{0, \text{dark}} \rangle$ and $\langle \Sigma_{0, \text{dark}} \rangle$ is given by Eq. (7.5). $\sigma_{\langle \Sigma_{0, \text{dark}} \rangle}$ is estimated from (7.1) as:

$$\sigma_{\langle \Sigma_{0, \text{dark}} \rangle} = 34.5 \ M_\odot \ pc^{-2}.$$  

(7.8)

The numerical value of the new $a_H$ parameter is:

$$a_H = 11.359 \pm 5.443 \ kpc.$$  

(7.9)

This result is similar to the value of $a_H$ presented in Eq. (4.18) at which it was calculated from the product $\rho_0 r_0$. The circular speed is given by Eq. (4.6) and the rotation curve of the simple model with $a_H$ given by (7.9) is presented in Fig. 7.1.
Figure 7.1: Rotation curve of the simple for $a_H$ calculated from $\langle \Sigma_{0,dark} \rangle$.

$a_H = 11.359 \text{ kpc}$ - solid line, for $a_H = 5.916 \text{ kpc}$ - dotted line,

for $a_H = 16.802 \text{ kpc}$ - dashed line. $R \in <0,100 > \text{ kpc}$

\section{7.3 Improved model ($\alpha = 0.174$)}

Now we calculate Gentile’s surface density for the model (2.1) with $\alpha = 0.174$. By using Eqs. (5.1) and (7.3) we get:

$$\langle \Sigma_{0,dark} \rangle = \frac{v_H^2}{\pi \kappa r_0^2} \int_0^{r_0} \left[ 1 - \frac{e^{-r^2/b^2} \left( 1 - \frac{2r^2}{b^2} \right)}{1 + \frac{e^{-r^2/b^2}}{a_H^2}} \right] dr . \tag{7.10}$$

Then from (7.10) we get:

$$\langle \Sigma_{0,dark} \rangle = \frac{v_H^2}{\pi \kappa r_0^2} \left[ r_0 + (\alpha - 1)r_0 e^{-\frac{r_0^2}{b^2}} - \alpha a_H \arctan \left( \frac{r_0}{a_H} \right) \right] = \frac{v_{circ}^2(r = r_0)}{\pi \kappa r_0} , \tag{7.11}$$

where the constants are presented in (5.7) and below Eq. (2.1). After enumeration of Eq. (7.11) we get the Gentile’s mean surface density of the improved model:

$$\langle \Sigma_{0,dark} \rangle = 3.27 \times 10^3 \ M_\odot \ pc^{-2} . \tag{7.12}$$
7.4 Finding new $\alpha$

We want to calculate the value of the $\alpha$ parameter from the Gentile’s mean surface density condition, which is presented in Eq. (2.4). Thus we create the model, which combines the models presented by Klačka (2009) and Gentile et al. (2009). The usage of Eqs. (7.11) and (2.4) yields

$$\langle \Sigma^{0}_{0, \text{dark}} \rangle = \frac{v^{2}_{H}}{\pi \kappa r_{0}^{2}} r_{0} + (\alpha - 1)r_{0}e^{-\frac{r^{2}_{0}}{b^{2}}} - \alpha a_{H} \arctan \left( \frac{r_{0}}{a_{H}} \right) = 72 \pm \frac{42}{27} M_{\odot} pc^{-2}. \quad (7.13)$$

We can not solve Eq. (7.13) by using analytic methods, because the scale-length depends on the $\alpha$ parameter. Thus we use numerical calculation to determine the value of $\alpha$. We derive the length-scale from Eqs. (5.1) and (5.2) as:

$$\rho(r = r_{0}) = \frac{\rho_{0}}{4} = \frac{v^{2}_{H}}{4 \pi \kappa} \left( \frac{3 - 3\alpha}{b^{2}} + \frac{\alpha}{a_{H}^{2}} \right), \quad (7.14)$$

also

$$\rho(r = r_{0}) = \frac{v^{2}_{H}}{4 \pi \kappa} \left[ \frac{1}{r_{0}^{2}} - \frac{\alpha}{1 + \frac{r_{0}}{a_{H}}} \frac{1}{r_{0}^{2}} + \frac{(\alpha - 1)e^{-\frac{r^{2}_{0}}{b^{2}}}}{r_{0}^{2}} \left( 1 - \frac{2r_{0}^{2}}{b^{2}} \right) \right]. \quad (7.15)$$

Eqs. (7.13)-(7.15) yield

$$\alpha = 0.00385 \pm 0.000225 \mp 0.00157., \quad (7.16)$$

which slightly differs from the value given by Eq. (6.8). The errors in Eq. (7.16) are estimated by using extremal values of $\langle \Sigma^{0}_{0, \text{dark}} \rangle$ ($\langle \Sigma^{0}_{0, \text{dark}} \rangle = 72 + 42 M_{\odot} pc^{-2}$ and $\langle \Sigma^{0}_{0, \text{dark}} \rangle = 72 - 27 M_{\odot} pc^{-2}$) in the calculations. The rotation curve of this model is presented in Fig. 7.2 and Fig. 7.3 for the values $\alpha = 0.00385, 0.00385 - 0.00157, 0.00385 + 0.00225$. The relation between $\alpha$ and the Gentile’s mean surface density is presented in Fig. 7.4.
Figure 7.2: Rotation curve of the new model (surface density) for $\alpha =$

- $0.00385$ - solid line, for $\alpha = 0.0024$ - dotted line, for $\alpha = 0.0061$ - dashed line. $R \in <0, 10 >$ kpc

Figure 7.3: Rotation curve of the new model (surface density) for $\alpha =$

- $0.00385$ - solid line, for $\alpha = 0.0024$ - dotted line, for $\alpha = 0.0061$ - dashed line. $R \in <10, 100 >$ kpc
Figure 7.4: The relation between $\alpha$ and the Gentile’s mean surface density

$<\Sigma_{0,\text{dark}}> [M_{\text{sun}} \text{ pc}^{-2}]$

$\Sigma_{0,\text{dark}}$ is the mean surface density in units of $M_{\text{sun}} \text{ pc}^{-2}$. The figure shows a linear relationship between $\alpha$ and $\Sigma_{0,\text{dark}}$. As $\alpha$ increases, $\Sigma_{0,\text{dark}}$ increases proportionally.
Chapter 8

The method of least squares

As we can see in previous chapters, the value of the $\alpha$ parameter may be calculated by using two various ways presented in chapters 6 and 7. Now we can combine both methods to calculate the $\alpha$ parameter more precisely. We use a variation of the method of least squares (MLS) to this goal.

By using Eqs. (6.1) and (7.1) we define a residual for each method. For Gentile’s surface density method we get:

$$res_1 = \langle \Sigma_{0,dark} \rangle - (72 \pm 42^{27}) \ M_\odot pc^{-2},$$

and the residual of the product $\rho_0 r_0$ method can be written:

$$res_2 = r_0 \rho_0 - (141 \pm 82^{52} \ M_\odot pc^{-2}).$$

The usage of Eqs. (7.13) and (8.1) yields

$$res_1 = \frac{v_H^2}{\pi \kappa r_0^2} \left[ r_0 + (\alpha - 1)r_0 e^{-\frac{r_0^2}{2\sigma^2}} - \alpha a_H \arctan \left( \frac{r_0}{a_H} \right) \right] - (72 \pm 42^{27}) \ M_\odot pc^{-2},$$

and by using Eqs. (5.2) and (8.2) we get:

$$res_2 = r_0 \frac{v_H^2}{4\pi \kappa} \left( \frac{3 - 3\alpha}{b^2} + \frac{\alpha}{a_H^2} \right) - (141 \pm 82^{52} \ M_\odot pc^{-2}),$$

where the scale-length is determined by Eq. (6.7) and numerical values of the constants are presented below Eq. (2.1).
CHAPTER 8. THE METHOD OF LEAST SQUARES

8.1 Simple MLS

We also define a sum of the squared residuals:

$$S = \sum_{i=1}^{2} (res)^2_i .$$  \hspace{1cm} (8.5)

We want to determine the $\alpha$ parameter by minimizing the sum of the squared residuals $S$. We numerically minimize the sum of the squared residuals by using Eqs.(8.3)-(8.5) and we calculate the numerical value of the $\alpha$ parameter as:

$$\alpha = 0.00339 \pm 0.00197 - 0.00126 .$$  \hspace{1cm} (8.6)

The rotation curve of this model is presented in Fig. 8.1 and Fig. 8.2

Figure 8.1: Rotation curve of the new model (the least squares) for $\alpha = 0.00339$ - solid line, for $\alpha = 0.00213$ - dotted line, for $\alpha = 0.00536$ - dashed line. $R \in < 0, 10 > kpc$
8.2 Weighted MLS

Now we consider that the Gentile’s surface density and the product $r_0 \rho_0$ have different errors. Thus we define the weights $w_i$ of the residuals (8.1) and (8.2):

$$w_i = \frac{1}{\sigma_i^2},$$

where $\sigma_i$ is the standard deviation. By using Eqs. (8.1) and (8.2) we estimate the standard deviation of the Gentile’s surface density and the product $r_0 \rho_0$. For the Gentile’s surface density we get:

$$\sigma_1 = 34.5 \ M_\odot pc^{-2}. \quad (8.8)$$

And for $r_0 \rho_0$ we get:

$$\sigma_2 = 67 \ M_\odot pc^{-2}. \quad (8.9)$$
(The ratio $\sigma_1/\sigma_2$ obtained from Eqs. (8.8)-(8.9) is equal to the ratios of 42/82 and 52/27 - see errors in Eqs. (8.1) and (8.2).) We also define a sum of the weighted squared residuals:

$$S_w = \sum_{i=1}^{2} w_i (res)^2_i.$$  \hspace{1cm} (8.10)

After numerical minimization of the sum of the weighted squared residuals by using Eqs. (8.3)-(8.4) and (8.7)-(8.10) we get the numerical value of the $\alpha$ parameter:

$$\alpha = 0.00353 \pm 0.00205 \pm 0.00131,$$  \hspace{1cm} (8.11)

which slightly differs from the result of the simple MLS. The rotation curve of this model is presented in Fig. 8.3 and Fig. 8.4.

Figure 8.3: Rotation curve of the new model (the weighted least squares)

for $\alpha = 0.00353$ - solid line, for $\alpha = 0.00222$ - dotted line, for $\alpha = 0.00558$ - dashed line. $R \in < 0, 10 > kpc$
Figure 8.4: Rotation curve of the new model (the weighted least squares) for $\alpha = 0.00353$ - solid line, for $\alpha = 0.00222$ - dotted line, for $\alpha = 0.00558$ - dashed line. $R \in <10,100> \text{ kpc}$

### 8.3 Comparison

The relative error of the $\alpha$ parameter calculated from simple MLS (8.6) and from weighted MLS (8.11):

$$\rho(\text{weightedMLS}) = 58.07\%,$$

$$\rho(\text{simpleMLS}) = 58.11\%.$$  \hspace{1cm} (8.12)

As we can see in Eq. (8.12), the weighted MLS leads to a slightly smaller relative error of the $\alpha$ parameter. It is in accordance with our expectation.
Chapter 9

Oort constants

The Oort constant are defined through the relations:

\[
A(R) = \frac{1}{2} \left[ \frac{v(R)}{R} - \frac{dv(R)}{dR} \right] \tag{9.1}
\]

and

\[
B(R) = -\frac{1}{2} \left[ \frac{v(R)}{R} + \frac{dv(R)}{dR} \right], \tag{9.2}
\]

for \( R = R_\odot \) (galactocentric distance of the Sun). \( A(R) \), \( B(R) \) are the Oort functions and \( v(R) \) is the circular speed. The values of the Oort constants for the solar region \( (R_\odot = 8 \text{ kpc}) \) of the individual model are presented in this chapter. The circular speed is given by Eq. (1.12):

\[
v(R) = \sqrt{v_{\text{bulge}}^2(R) + v_{\text{disk}}^2(R) + v_{\text{halo}}^2(R)} \tag{9.3}
\]

\( v_{\text{bulge}}(R) \) and \( v_{\text{disk}}(R) \) are given by Eqs. (1.8) and (1.10) and \( v_{\text{halo}}(R) \) depends on the model of the galactic halo.

9.1 Simple model

The simple model of the galactic halo is discussed in chapter 4 and the circular speed \( \text{(9.4)} \) is:

\[
v_{\text{circ}}^2(r) = v_H^2 \left[ 1 - \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) \right], \tag{9.4}
\]

where \( v_H = 220 \text{ km/s} \). By using Eqs. (1.8), (1.10) and (9.3)-(9.4) we get the circular speed of the Galaxy. By using Eqs. (9.1) and (9.2) we calculated the Oort constants for the solar region.
CHAPTER 9. OORT CONSTANTS

For $a_H = 0.04383 \text{ kpc}$:

$$A(R = 8 \text{ kpc}) = 19.02 \text{ km.s}^{-1}\text{kpc}^{-1},$$

$$B(R = 8 \text{ kpc}) = -17.01 \text{ km.s}^{-1}\text{kpc}^{-1},$$

(9.5)

and for $a_H = 11 \pm 5.227 \text{ kpc}$ calculated in the section 4.4.1:

$$A(R = 8 \text{ kpc}) \in (12.662, 12.845) \text{ km.s}^{-1}\text{kpc}^{-1},$$

$$B(R = 8 \text{ kpc}) \in (-14.985, -11.760) \text{ km.s}^{-1}\text{kpc}^{-1}.$$  (9.6)

9.2 Improved model

The circular speed of the galactic halo of the improved model (2.1) is

$$v_{circ}^2 = v_H^2\left(1 - \alpha \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) - (1 - \alpha) \exp \left( -\frac{r^2}{b^2} \right) \right),$$

(9.7)

where $v_H = 220 \text{ km/s}$, $b = 37.3760 \text{ kpc}$ and $\alpha = 0.174$. By using Eqs. (1.8), (1.10), (9.3) and (9.7) we get the circular speed of the Galaxy. By using Eqs. (9.1) and (9.2) we calculated the Oort constants for the solar region:

$$A(R = 8 \text{ kpc}) = 14.19 \text{ km.s}^{-1}\text{kpc}^{-1},$$

$$B(R = 8 \text{ kpc}) = -12.40 \text{ km.s}^{-1}\text{kpc}^{-1}.$$  (9.8)

9.3 Model with the new $\alpha$ parameter

The new model of the galactic halo is discussed in chapters 6 - 8 and the circular speed is given by Eq. (9.7), where $v_H = 220 \text{ km/s}$, $a_H = 0.04383 \text{ kpc}$, $b = 37.3760 \text{ kpc}$. The $\alpha$ parameter is given by Eq. (8.11):

$$\alpha = 0.00353 \pm 0.00205 \pm 0.00131.$$  (9.9)

By using Eqs. (1.8), (1.10), (9.3) and (9.7) we get the circular speed of the Galaxy. By using Eqs. (9.1) and (9.2) we calculated the Oort constants for the solar region:

$$A(R = 8 \text{ kpc}) \in (12.961, 12.979) \text{ km.s}^{-1}\text{kpc}^{-1},$$

$$B(R = 8 \text{ kpc}) \in (-11.202, -11.224) \text{ km.s}^{-1}\text{kpc}^{-1}.$$  (9.10)
The relation between the Oort constants in the solar region and the $\alpha$ parameter is presented in Fig. 9.1.

![Figure 9.1: The relation between the Oort constants and the $\alpha$ parameter for the solar region ($R_\odot = 8\text{ kpc}$). Oort constant A - solid line, Oort constant B - dashed line.]

9.4 Discussion

The values given by Eqs. (9.8) are consistent with the most probable values $A(R_\odot)$ and $B(R_\odot)$ presented by (Klačka 2009) (1.7). The values presented by Eqs. (9.6) and Eqs. (9.10) are not consistent with $A(R_\odot)$ and $B(R_\odot)$. 
Chapter 10

The final model

We calculated a new value of the $\alpha$ parameter in the previous chapters. The new model of the galactic halo is given by the circular speed (2.1) and the $\alpha$ parameter (8.11). This model satisfies our primary goal presented in chapter 9. As we can see in Fig. 8.3 and Fig. 8.4 it produces the flat rotation curve.

The value of the $\alpha$ parameter was calculated on the basis of the current models of the dark matter halo. This is represented with Eqs. (2.2) and (2.4) which are presented by Donato et al. (2009) and Gentile et al. (2009), respectively. In this chapter we will use the $\rho_0 r_0$ condition presented by Donato et al. (2009):

$$r_0 \rho_0 = 141 \pm_{52}^{82} M_\odot pc^{-2}. \quad (10.1)$$

However, the model with the new value of the $\alpha$ parameter does not satisfy secondary goal presented in chapter 9. As we can see in the previous chapter 9 it does not give the realistic values of the Oort constants in the solar region (1.7). Thus we will calculate all free parameters in this model ($\alpha$, $a_H$, $b$) in order to satisfy this condition, too. We do not consider the errors in Eq. (10.1) for the significant numerical difficulties of the calculations. We use only (10.1) and we do not use Gentile's condition (2.4) for the same reason.

10.1 The method of least squares

We define a residual for each condition. For the $\rho_0 r_0$ product we get:

$$res_{\rho_0r_0} = r_0 \rho_0 - (141 \ M_\odot pc^{-2}) , \quad (10.2)$$
where \( r_0 \rho_0 \) of our model is given by Eqs. (5.2) and (6.7). For the Oort constant \( A \):

\[
res_A = A(R = R_\odot) - 14.2 \text{ km.s}^{-1}\text{kpc}^{-1},
\]  

(10.3)

where \( A(R = R_\odot) \) is given by Eq. (9.1) and \((14.2 \text{ km.s}^{-1}\text{kpc}^{-1})\) is the realistic value of the A Oort constant in the solar region (1.7). And for the Oort constant \( B \):

\[
res_B = B(R = R_\odot) + 12.4 \text{ km.s}^{-1}\text{kpc}^{-1},
\]  

(10.4)

where \( B(R = R_\odot) \) is given by Eq. (9.2) and \((-12.4 \text{ km.s}^{-1}\text{kpc}^{-1})\) is the realistic value of the B Oort constant in the solar region (1.7). By using Eqs. (1.8), (1.10), (9.3) and (9.7) we get the circular speed of the Galaxy which is used in Eqs. (9.1) and (9.2). We define the sum of the squared residuals:

\[
S = (res_\rho_0 r_0)^2 + (res_A)^2 + (res_B)^2.
\]  

(10.5)

We determine the parameters \( \alpha \), \( a_H \) and \( b \) by minimizing the sum of the squared residuals (10.5). After numerical minimization we get the values of all free parameters of our model:

\[
\begin{align*}
\alpha &= 0.184 \pm 0.002, \\
a_H &= (2.156 \pm 0.002) \text{ kpc}, \\
b &= (54.5 \pm 0.5) \text{ kpc},
\end{align*}
\]  

(10.6)

the errors are caused by the numerical minimization.

### 10.2 Properties of the final model

The new model of the galactic halo is given by Eqs. (2.1), (10.6) and \( v_H = 220 \text{ km.s}^{-1} \).

#### 10.2.1 Surface density

The surface density of the galactic model is given by Eq. (5.5). The scale-length for this final model is

\[
r_0 = 3 895.68 \text{ pc},
\]  

(10.7)
if also Eq. (6.7) is used. By using Eqs. (5.6), (10.6)–(10.7) we get the mean surface density

\[ \langle \Sigma \rangle_0 = 2.52 \times 10^2 \, M_\odot \, pc^{-2} . \tag{10.8} \]

As we can see by comparison of Eqs. (5.9) and (10.8), the value of the mean surface density of the final model is approximately 30 times smaller than the mean surface density of the improved model with \( \alpha = 0.174 \).

10.2.2 Donato et al. (2009) - \( \rho_0 r_0 \)

The \( \rho_0 r_0 \) product is given by Eqs. (5.2), (6.7) and (10.7) and for the final model it reaches:

\[ r_0 \rho_0 = 141 \, M_\odot pc^{-2} , \tag{10.9} \]

which is in accordance with condition (2.2) presented by Donato et al. (2009).

10.2.3 Gentile et al. (2009) - surface density

Gentile's surface density is given by Eq. (7.11) and the values of the constants of the final model are given by Eqs. (10.6)–(10.7) and \( v_H = 220 \, km/s \). The numerical value of the Gentile's surface density of the final model is:

\[ \langle \Sigma_{0,\text{dark}} \rangle = 73.28 \, M_\odot \, pc^{-2} . \tag{10.10} \]

This value is in accordance with condition (2.4) presented by Gentile et al. (2009).

10.2.4 Rotation curve

The circular speed of the galactic halo of the final model is given by Eqs. (2.1), (10.6) and \( v_H = 220 \, km/s \). The rotation curve is presented in Fig. 10.1.
The circular speed of the Galaxy is given by Eqs. (1.13) and the values of the constants are
10.2.5 The Oort constants

By using Eqs. (10.1) and (10.2) we calculated the value of the Oort constants for the solar region \( R_\odot = 8 \text{ kpc} \), where the circular speed \( v(R) \) is given by Eqs. (10.3) and the values of the constants are given by Eqs. (10.10), (10.4), and \( v_H = 220 \text{ km/s}, R_d = 3.5 \text{ kpc} \). The numerical values of the Oort constants of the final model are:

\[
A(R = 8 \text{ kpc}) = 13.6 \text{ km.s}^{-1}.\text{kpc}^{-1}, \\
B(R = 8 \text{ kpc}) = -11.93 \text{ km.s}^{-1}.\text{kpc}^{-1}.
\] (10.11)

We assume the most probable values of the Oort constants presented by Kláčka (2009) (1.7) as:

\[
A(R = 8 \text{ kpc}) \in (13.7, 14.7) \text{ km.s}^{-1}.\text{kpc}^{-1}, \\
B(R = 8 \text{ kpc}) \in (-11.9, -12.9) \text{ km.s}^{-1}.\text{kpc}^{-1}.
\] (10.12)

After comparison of our and the most probable values of the Oort constants we see, that our value of the Oort constant \( A \) is not fully consistent with the most probable value, but the difference is almost negligible. Our value of the Oort constant \( B \) is in accordance with the most probable value.

10.3 Discussion

We created a model of the galactic halo which satisfies all goals set out in chapter 11. It combines the models presented by Kláčka (2009) (for our Galaxy) and Donato et al. (2009) (for other galaxies).

The final model produces the flat rotation curve (see Fig. 10.2), it yields the realistic values of the Oort constants (see Eqs. (10.11)-(10.12)) and it is consistent with the conditions presented by Donato et al. (2009) and Gentile et al. (2009) (see Eqs. (10.9) and (10.10)).
Chapter 11

The Solar motion

This chapter contains the contribution presented in Student Science Conference (2010), Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava (Nagy and Cayao 2010) and in Czecho-Slovak Student Science Conference in physics (2010). Several results are taken from the paper in preparation: Klačka J., Nagy R., Cayao J., Kómár L., Pástor P., Jurčí M.: The solar motion.

11.1 Introduction

"The kinematics of stars near to the Sun has long been known to provide crucial information regarding both the structure and evolution of the Milky Way" (Dehnen and Binney 1998). That is the reason why we want to calculate the solar motion in the reference frame connected with the nearest stars. We identify series of \( N \) stars with the heliocentric distances less than 100, 40, 15 pc and then determine the velocity of the Sun relative to the mean velocity of these stars.

"The mean motion of all stars in the volume element considered is clearly the physically most meaningful when considered in the framework of the Galaxy. A point possessing this motion defines what is known as the Local Standard of Rest (LSR)" (Mihalas and McRae-Routly 1968). The LSR is a point in space which has a galactic velocity equal to the average velocity of stars in the solar neighborhood, including the Sun.
11.2 The solar motion

We consider a set of \( N \) stars. The coordinates of \( i \)-star in the equatorial coordinate system are
\[
x_i = r_i \cos \alpha_i \cos \delta_i, \quad y_i = r_i \sin \alpha_i \cos \delta_i, \quad z_i = r_i \sin \delta_i,
\]
where \( r_i \) is the heliocentric distance, \( \alpha_i \) is the right ascension and \( \delta_i \) is the declination of the \( i \)-star. The velocity of the \( i \)-star with respect to the Sun is
\[
\vec{v}_i = \vec{V}_i - \vec{V}_S, \tag{11.1}
\]
where \( \vec{V}_i \) and \( \vec{V}_S \) are the velocities of the \( i \)-star and the Sun in the frame of the LSR. The velocities \( \vec{V}_i, \vec{V}_S \) and \( \vec{v}_i \) can be written as:
\[
\vec{V}_S = (X_S, Y_S, Z_S), \quad \vec{V}_i = (X_i, Y_i, Z_i), \tag{11.2}
\]
and,
\[
\begin{align*}
v_{x,i} &= \dot{r}_i \cos(\alpha_i) \cos(\delta_i) - 4.74 r_i \left[ (\mu_{\alpha,i} \cos(\delta_i)) \sin(\alpha_i) \\
&\quad + \mu_{\delta,i} \sin(\alpha_i) \sin(\delta_i) \right], \\
v_{y,i} &= \dot{r}_i \sin(\alpha_i) \cos(\delta_i) + 4.74 r_i \left[ (\mu_{\alpha,i} \cos(\delta_i)) \cos(\alpha_i) \\
&\quad - \mu_{\delta,i} \sin(\alpha_i) \sin(\delta_i) \right], \\
v_{z,i} &= \dot{r}_i \sin(\delta_i) + 4.74 r_i \mu_{\delta,i} \cos(\delta_i), \tag{11.3}
\end{align*}
\]
where \( \dot{r}_i \) is the radial velocity and \( \mu_{\alpha,i}, \mu_{\delta,i} \) are the proper motions in right ascension and declination. Since the units used in our calculations are
\[
[v] = [\dot{r}] = km \, s^{-1}, \quad [r] = pc, \quad [\mu_\alpha] = [\mu_\delta] = " \, yr^{-1},
\]
the numerical factor in Eq. (11.3) is 4.74.

Proper motions, radial velocities, calculated from the redshift, and heliocentric distances, calculated from the parallax are the observational data and can be used to describe the solar motion.

Few ways to calculate the solar motion in respect of the LSR exist. If we have all parameters of stars in our set, we can find the solar motion using direct calculation. On the other hand, if some of the parameters are unknown, we can approximate the solution using the least square method.
11.2.1 Determining \( \vec{V}_S \) by direct calculation

The LSR is given by:

\[
\sum_{i=1}^{N} \vec{V}_i = 0,
\]

(11.5)

where \( \vec{V}_i \) is given by Eq. (11.2). The average value of Eq. (11.1) is:

\[
\frac{1}{N} \sum_{i=1}^{N} \vec{v}_i = \frac{1}{N} \sum_{i=1}^{N} \vec{V}_i - \vec{V}_S.
\]

(11.6)

Then rewriting Eq. (11.6) by using Eq. (11.5) we get:

\[
\frac{1}{N} \sum_{i=1}^{N} \vec{v}_i = -V_s,
\]

(11.7)

or:

\[
X_S = -\frac{1}{N} \sum_{i=1}^{N} \tilde{r}_i \cos(\alpha_i) \cos(\delta_i) + 4.74 \frac{1}{N} \sum_{i=1}^{N} r_i \mu_{\alpha,i} \sin(\alpha_i) \cos(\delta_i)
\]

\[+4.74 \frac{1}{N} \sum_{i=1}^{N} r_i \mu_{\delta,i} \cos(\alpha_i) \sin(\delta_i),
\]

\[
Y_S = -\frac{1}{N} \sum_{i=1}^{N} \tilde{r}_i \sin(\alpha_i) \cos(\delta_i)
\]

\[ -4.74 \frac{1}{N} \sum_{i=1}^{N} r_i \mu_{\alpha,i} \cos(\alpha_i) \cos(\delta_i)
\]

\[+4.74 \frac{1}{N} \sum_{i=1}^{N} r_i \mu_{\delta,i} \sin(\alpha_i) \sin(\delta_i),
\]

\[
Z_S = -\frac{1}{N} \sum_{i=1}^{N} \tilde{r}_i \sin(\delta_i) - 4.74 \frac{1}{N} \sum_{i=1}^{N} r_i \mu_{\delta,i} \cos(\delta_i).
\]

(11.8)

Previous equations uniquely determine the solar motion in respect of the LSR.

11.2.2 Determining \( \vec{V}_S \) by Least square method

It is not immediately obvious from equations (11.1) and (11.3) how \( \vec{V}_S \) should be determined from a given body of data. We address this problem for the case of radial velocities and proper motions data. By using the Least square method we find \( \vec{V}_S \) as a generalization of the results
presented by (Mihalas and McRae-Routly 1968, pp. 88-103). Let 3N orthogonal unit vectors

\( (i = 1, \cdots, N) \) are:

\[
\begin{align*}
\vec{e}_{r,i} &= (\cos(\alpha_i) \cos(\delta_i), \sin(\alpha_i) \cos(\delta_i), \sin(\delta_i)) \\
\vec{e}_{\alpha,i} &= (-\sin(\alpha_i), \cos(\alpha_i), 0) \\
\vec{e}_{\delta,i} &= (-\cos(\alpha_i) \sin(\delta_i), -\sin(\alpha_i) \sin(\delta_i), \cos(\delta_i)),
\end{align*}
\]

where \( \vec{e}_{r,i} \) are radial vectors and \( \vec{e}_{\alpha,i}, \vec{e}_{\delta,i} \) are vectors in the direction of the right ascension and the declination of the \( i \)-star. We define a general unit vector:

\[
\vec{e}_i = \vec{e}_{\alpha,i} \cos \phi \sin \theta + \vec{e}_{\delta,i} \sin \phi \sin \theta + \vec{e}_{r,i} \cos \theta,
\]

\( \phi \in (0, 2\pi), \theta \in (0, \pi) \).

The coordinate of \( \vec{v}_i \) in the direction of the vector \( \vec{e}_i \) is given by \( \vec{v}_i.e_i = (V_i - V_S).e_i \). Therefore, by using Eqs. (11.3) we get:

\[
(\vec{V}_i - \vec{V}_S).\vec{e}_i = \dot{r}_i \cos \theta + 4.74r_i(\mu_{\alpha,i} \cos(\delta_i) \cos \phi + \mu_{\delta,i} \sin \phi) \sin \theta .
\]

We know that:

\[
(\vec{V}_i - \vec{V}_S).e_i = (X_i - X_S)(-\sin(\alpha_i) \cos \phi \sin \theta - \cos(\alpha_i) \sin(\delta_i) \cos \phi \sin \theta + \cos(\alpha_i) \cos(\delta_i) \cos \theta) \\
+ (Y_i - Y_S)(\cos(\alpha_i) \cos \phi \sin \theta - \sin(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta + \sin(\alpha_i) \cos(\delta_i) \cos \theta) \\
+ (Z_i - Z_S)(\cos(\delta_i) \sin \phi \sin \theta + \sin(\delta_i) \cos \theta).
\]

From equations (11.11) and (11.12) follows

\[
\dot{r}_i \cos \theta + 4.74r_i(\mu_{\alpha,i} \cos(\delta_i) \cos \phi + \mu_{\delta,i} \sin \phi) \sin \theta \\
- (X_i - X_S)(-\sin(\alpha_i) \cos \phi \sin \theta - \cos(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta + \cos(\alpha_i) \cos(\delta_i) \cos \theta) \\
+ (Y_i - Y_S)(\cos(\alpha_i) \cos \phi \sin \theta - \sin(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta + \sin(\alpha_i) \cos(\delta_i) \cos \theta) \\
- (Z_i - Z_S)(\cos(\delta_i) \sin \phi \sin \theta + \sin(\delta_i) \cos \theta) \equiv p_i,
\]

where

\[
\begin{align*}
p_i &= (\mu_{\alpha,i} \cos(\delta_i) \cos \phi + \mu_{\delta,i} \sin \phi) \sin \theta \\
&\quad \text{and} \quad \mu_{\alpha,i}, \mu_{\delta,i} \equiv \mu_i.
\end{align*}
\]
where $p_i$ is the residual and $p_i = 0$, exactly. But we admit $p_i \neq 0$ and we use the Least square method to find $X_S, Y_S, Z_S$. Defining the sum of the squared residuals

$$S(X_S, Y_S, Z_S) = \sum_{i=1}^{N} [p_i]^2,$$

we minimize it:

$$\frac{\partial S}{\partial X_S} = \frac{\partial S}{\partial Y_S} = \frac{\partial S}{\partial Z_S} = 0. \tag{11.15}$$

We assume

$$\langle \vec{V}_i \cos^k(\alpha_i) \sin^l(\alpha_i) \cos^m(\delta_i) \sin^n(\delta_i) \rangle = 0 \tag{11.16}$$

for arbitrary $k, l, m, n$ (Mihalas and McRae-Routly 1968, p.95). We denote

$$\Gamma_i \equiv \dot{r}_i \cos \phi + 4.74 r_i (\mu_{\alpha,i} \cos(\delta_i) \cos \phi$$

$$+ \mu_{\delta,i} \sin \phi) \sin \theta + X_S (\sin(\alpha_i) \cos \phi \sin \theta$$

$$- \cos(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta + \cos(\alpha_i) \cos(\delta_i) \cos \theta$$

$$+ Y_S (\cos(\alpha_i) \cos \phi \sin \theta$$

$$- \sin(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta$$

$$+ \sin(\alpha_i) \cos(\delta_i) \cos \theta$$

$$+ Z_S (\cos(\delta_i) \sin \phi \sin \theta$$

$$+ \sin(\delta_i) \cos \theta) \tag{11.17}.$$}

Then rewriting Eq. (11.15) by using Eqs. (11.16)-(11.17) we get:

$$\frac{\partial S}{\partial X_S} = 2 \sum_{i=1}^{N} \left[ \Gamma_i (\sin(\alpha_i) \cos \phi \sin \theta$$

$$- \cos(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta$$

$$+ \cos(\alpha_i) \cos(\delta_i) \cos \theta) \right] = 0, \tag{11.18}$$

$$\frac{\partial S}{\partial Y_S} = 2 \sum_{i=1}^{N} \left[ \Gamma_i (\cos(\alpha_i) \cos \phi \sin \theta$$

$$- \sin(\alpha_i) \sin(\delta_i) \sin \phi \sin \theta$$

$$+ \sin(\alpha_i) \cos(\delta_i) \cos \theta) \right] = 0,$$

$$\frac{\partial S}{\partial Z_S} = 2 \sum_{i=1}^{N} \left[ \Gamma_i (\cos(\delta_i) \sin \phi \sin \theta$$

$$+ \sin(\delta_i) \cos \theta) \right] = 0.$$

$\phi, \theta$ are arbitrary angles ($\langle 0, 2\pi \rangle, \theta \in \langle 0, \pi \rangle$). Therefore, coefficients of the same combination of ($\sin \phi, \cos \phi, \sin \theta, \cos \theta$) are equal to zero in each equation in (11.18). This leads to the six
CHAPTER 11. THE SOLAR MOTION

independent linear systems of three equations in unknowns \((X_S, Y_S, Z_S)\). These systems are used to determine the solar motion and they are presented in appendix A ((A.1)-(A.6)).

The equations (A.1) represent the way how to calculate the solar motion by using only the proper motions in the right ascension. However, they allow us to determine the solar motion only in \(X_S, Y_S\) directions. Thus we do not consider these equations in our calculations. The system of equations (A.2) depends on the proper motions both in the right ascension and the declination. Eqs. (A.3) contain the radial velocities and the proper motions in the right ascension and the declination. The system (A.4) depends on the radial velocities and the proper motion in the right ascension. The equations (A.5) contain only the proper motion in the declination and the system (A.6) depends only on the radial velocities. Eqs. (A.5) and (A.6) are also presented in Mihalas and McRae-Routly (1968).

The system of equations (A.0.1) is similar to the type of Eqs. (5-13)-(5-14) in (Mihalas and McRae-Routly 1968, p. 97). The system of equations (A.0.5) is similar to the type of Eqs. (5-10)-(5-12) in (Mihalas and McRae-Routly 1968, p. 97). Surprising is the following formulation in (Mihalas and McRae-Routly 1968, p. 97): "Finally, if we add equations (5-10) to (5-13) and (5-11) to (5-14) to utilize the proper motion information as fully as possible, we obtain three equations of their form ..." (equations for \(X_S, Y_S, Z_S\)). However, the access of the authors yield the result which is not consistent with our system (A.0.2).

The systems (A.2)- (A.6) provide a complete information about the solar motion \((X_S, Y_S, Z_S)\). Thus we determine the solar motion for each system.

11.3 The Equatorial and Galactic Coordinates

Up to now we have used equatorial coordinate system with the right ascension \(\alpha\) and the declination \(\delta\). Dealing with the motions in the Galaxy, we will use a galactic coordinate system with the galactic longitude \(l\) and the galactic latitude \(b\).

Eqs. (11.3), (11.8) - (11.18), are given in the equatorial coordinates \((\alpha, \delta)\). We need to transform them to the galactic coordinates \((l,b)\). If we want to avoid transformation of the proper motions, we can use the fact that components of the solar motion velocity are \(X_S = V_S \cos \alpha \cos \delta, Y_S = V_S \sin \alpha \cos \delta, Z_S = V_S \sin \delta, V^2 = X^2_S + Y^2_S + Z^2_S\) in terms of the equatorial coordinates, and, \((X_S)_{gal} = V_S \cos l \cos b, (Y_S)_{gal} = V_S \sin l \cos b, (Z_S)_{gal} = V_S \sin b\) in terms
CHAPTER 11. THE SOLAR MOTION

of the galactic coordinates. The angles \( l \) and \( b \) can be obtained from

\[
\begin{align*}
\cos b \cos (l - l_0) &= \cos \delta \sin (\alpha - \alpha_0), \\
\cos b \sin (l - l_0) &= \sin \delta \cos \delta_0 \\
&- \cos \delta \sin \delta_0 \cos (\alpha - \alpha_0), \\
\sin b &= \sin \delta \sin \delta_0 \\
&+ \cos \delta \cos \delta_0 \cos (\alpha - \alpha_0),
\end{align*}
\]

(11.19)

where \( l_0 = 33.932^\circ \), \( \alpha_0 = 192.85948^\circ \), and \( \delta_0 = 27.17825^\circ \) for the Epoch J2000.0.

11.4 Results

We selected stars from the database SIMBAD \((SIMBAD Astronomical Database 2010)\). At first, we chose stars with the heliocentric distance less than 100pc, but only stars with complete observational data (radial velocities, proper motion, parallax). The number of stars to that distance was 24,167. Then we selected stars with the best quality index (A) in the parallax, the proper motions and the radial velocities. This selection reduced the number of stars to 769. These stars are used in our calculations of the solar motion. Remember that equation (11.2) is given in the equatorial coordinates, but the results are shown in the galactic coordinates.

11.4.1 For \( r_i < 100 \text{pc} \)

For the set of 769 stars with \( r_i < 100 \text{pc} \) we computed the solar motion which is presented in Table \( 11.1 \).

We can see in the Table \( 11.1 \) the results \((Z_S, |\vec{V}_S|)\) from the direct method (without approximations) for this case are in accord with the standard solar motion presented by (Mihalas and McRae-Routly 1968, pp. 88-103) (see Table \( 11.4 \)). The solar motions calculated from Eq. (A.5) and (A.6) are similar to the results of the direct method. The rest of equations deduced from the Least square method ((A.2)-(A.4)) give quite different solutions (see Table \( 11.1 \)) than the direct calculation. Thus these equations are useless to determine the solar motion.


Table 11.1: The solar motion in the galactic coordinates for our sample of stars with $r_i < 100$ pc calculated using the direct method (Eqs. (11.8)) and the Least square method (Eqs. (A.2)-(A.6)).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Method} & X_S \text{ [km/s]} & Y_S \text{ [km/s]} & Z_S \text{ [km/s]} \\
\hline
\text{Eq. (11.8)} & 7.63 & 16.65 & 7.43 & 19.76 \\
\text{Eq. (A.2)} & 74.7 & -34.0 & 159.5 & 179.4 \\
\text{Eq. (A.3)} & 76.2 & 106.6 & -0.7 & 131.1 \\
\text{Eq. (A.4)} & 59.3 & -53.4 & 38.5 & 88.6 \\
\text{Eq. (A.5)} & 8.82 & 17.64 & 8.07 & 21.31 \\
\text{Eq. (A.6)} & 10.59 & 17.86 & 6.90 & 21.88 \\
\hline
\end{array}
\]

11.4.2 For $r_i < 40$ pc

Here we consider stars with $r_i < 40$ pc. This set contains 360 stars. Calculated values of the solar motion in the galactic coordinates are shown in Table 11.2.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Method} & X_S \text{ [km/s]} & Y_S \text{ [km/s]} & Z_S \text{ [km/s]} & |V_S| \text{ [km/s]} \\
\hline
\text{Eq. (11.8)} & 7.09 & 18.53 & 7.70 & 21.28 \\
\text{Eq. (A.2)} & 11.5 & 10.0 & 50.5 & 52.8 \\
\text{Eq. (A.3)} & 150.9 & -10.7 & -49.1 & 159.1 \\
\text{Eq. (A.4)} & 1678.8 & -1314.3 & -142.4 & 2136.9 \\
\text{Eq. (A.5)} & 5.92 & 18.28 & 10.53 & 21.91 \\
\text{Eq. (A.6)} & 13.18 & 20.95 & 4.37 & 25.13 \\
\hline
\end{array}
\]

Table 11.2: The solar motion in the galactic coordinates for our set of stars with $r_i < 40$ pc calculated using the direct method (Eqs. (11.8)) and the Least square method (Eqs. (A.2)-(A.6)).

The solutions of equations (A.2) and (A.4) are quite different from the solar motion calculated by the direct method again. The equations (A.5) give correct absolute value of the velocity, but the components of the solar motion are slightly different from our direct calculations. The solution of Eq. (A.6) is more different than (A.5) solution, but it is still good estimation of the solar motion.
11.4.3 For \( r_i < 15\text{pc} \)

The nearest hundred stars in the solar neighborhood have the heliocentric distance less than 15pc. Our results are presented in Table 11.3. The solutions of Eq. (A.2)-(A.4) are useless

| Method    | \(X_S \) [\(\text{km/s}\)] | \(Y_S \) [\(\text{km/s}\)] | \(Z_S \) [\(\text{km/s}\)] | \(|V| \) [\(\text{km/s}\)] |
|-----------|-----------------|-----------------|-----------------|-----------------|
| Eq. (11.8) | 14.16           | 17.59           | 7.09            | 23.68           |
| Eq. (A.2)  | -8.3            | 15.7            | 30.4            | 35.2            |
| Eq. (A.3)  | 120.0           | -43.7           | 25.3            | 130.2           |
| Eq. (A.4)  | 41.77           | -56.5           | 20.75           | 73.3            |
| Eq. (A.5)  | 7.02            | 18.42           | 2.37            | 19.85           |
| Eq. (A.6)  | 24.08           | 20.58           | 3.77            | 31.9            |

Table 11.3: The solar motion in the galactic coordinates for our set of stars with \( r_i < 15\text{pc} \) calculated using the direct method (Eqs. (11.8)) and the Least square method (Eqs. (A.2)-(A.6)).

in comparison with the direct method (Eq. (11.8)). The equations (A.5) give relatively good approximation of the solar motion in respect of this LSR. But the solution of equations (A.6) differ from the direct method (Eq. (11.8)), especially in the absolute value of the velocity and \(Z_S\) component. However, these results may be influenced by using small Solar neighborhood.

11.4.4 The solution of Eqs. (11.8) for different distances

Now we use the direct method (Eqs. (11.8)) to calculate the solar motion for the Solar neighborhoods with various radii. The results are presented in Table 11.4.

The relation between the velocity components of the solar motion and the radius of the solar neighborhood is illustrated in Fig. (11.1).
### Table 11.4: The solar motion calculated by direct method (Eqs. (11.8)) for different distances.

| Distance [pc] | $X_S$ [km/s] | $Y_S$ [km/s] | $Z_S$ [km/s] | $|V_S|$ [km/s] | Number of stars |
|---------------|--------------|--------------|--------------|--------------|-----------------|
| 10            | 12.57        | 15.36        | 6.14         | 20.77        | 44              |
| 20            | 12.10        | 17.51        | 6.98         | 22.41        | 165             |
| 30            | 8.18         | 17.41        | 7.28         | 20.57        | 273             |
| 40            | 7.08         | 18.53        | 7.70         | 21.28        | 360             |
| 50            | 9.34         | 18.38        | 7.59         | 21.97        | 445             |
| 60            | 9.11         | 18.03        | 7.61         | 21.59        | 532             |
| 70            | 8.53         | 18.09        | 7.44         | 21.34        | 603             |
| 80            | 7.91         | 17.52        | 7.64         | 20.69        | 669             |
| 90            | 7.57         | 16.99        | 7.43         | 20.04        | 725             |
| 100           | 7.63         | 16.65        | 7.43         | 19.77        | 769             |

Figure 11.1: The relation between the velocity components of the solar motion and radius of the solar neighborhood.
11.5 Application

11.5.1 The solar oscillation – simple access

The knowledge of the solar motion, especially $Z_S$ component, allows us to calculate the solar oscillations in direction perpendicular to the galactic equator. In this case, we can use solar equation of motion presented by Klačka (2009):

$$\ddot{z} = -[4\pi G \rho + 2(A^2 - B^2)]z$$  \hspace{1cm} (11.20)

where $A, B$ are the Oort constants and $\rho$ is the mass density in the galactic equatorial plane in a galactocentric distance equal to the galactocentric distance of the Sun. The values of the constants are:

$$A = 14.2 \text{ km} \text{s}^{-1} \text{kpc}^{-1},$$
$$B = -12.4 \text{ km} \text{s}^{-1} \text{kpc}^{-1},$$
$$\rho = 0.13 M_\odot \text{pc}^{-3}.$$

Eq. (11.20) is the equation of motion of a linear harmonic oscillator with angular frequency $\omega$. In our case $\omega^2 = [4\pi G \rho + 2(A^2 - B^2)]$. The solution of Eq. (11.20) is:

$$z = C_1 \cos(\omega t) + C_2 \sin(\omega t),$$
$$C_1 = z(t = 0),$$
$$C_2 = \frac{\dot{z}(t = 0)}{\omega},$$

(11.21)

$$z_{\text{max}} = \sqrt{[z(t = 0)]^2 + \left[\frac{\dot{z}(t = 0)}{\omega}\right]^2},$$

since the constants $C_1, C_2$ are given by initial conditions. Our results of the solar motion in $z$–direction are used for various initial values of $\dot{z}(t = 0)$ and for $z(t = 0) = 30 \text{pc}$.

As we can see in Table[11.5], the maximal distance of the Sun from the galactic equator markedly depends on the $z$–component of the solar motion.

11.5.2 The solar oscillation – improved access

Using $Z_S$ component and a better approximation of a mass density as a function of the coordinate $z$, $\rho = \rho_{\text{disk}} (1 - u |z|) + \rho_{\text{halo}}, u = 3.3 \text{ kpc}^{-1}$ (Klačka 2009), numerical calculation of the solar oscillations in the direction perpendicular to the galactic equator yields the results summarized in Table[11.6].
Table 11.5: Amplitude \( z_{max} \) and periods of the solar oscillations with the initial position \( z(t = 0) = 30 \text{pc} \). Various initial velocities \( Z_S \) are used.

<table>
<thead>
<tr>
<th>( Z_S [\text{km/s}] )</th>
<th>( z_{max} [\text{pc}] )</th>
<th>( P [\text{Myrs}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>min. value of ( Z_S )</td>
<td>6.14</td>
<td>78.71</td>
</tr>
<tr>
<td>max. value of ( Z_S )</td>
<td>7.70</td>
<td>96.06</td>
</tr>
<tr>
<td>average ( Z_S ) value</td>
<td>7.32 ± 0.15</td>
<td>91.79 ± 1.66</td>
</tr>
</tbody>
</table>

Table 11.6: Amplitude \( z_{max} \) and periods of the solar oscillations with the initial position \( z(t = 0) = 30 \text{pc} \). Various initial velocities \( Z_S \) are used. Mass density as a function of \( z \) is considered.

<table>
<thead>
<tr>
<th>( Z_S [\text{km/s}] )</th>
<th>( z_{max} [\text{pc}] )</th>
<th>( P [\text{Myrs}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimum value of ( Z_S )</td>
<td>6.14</td>
<td>81.8</td>
</tr>
<tr>
<td>maximum value of ( Z_S )</td>
<td>7.70</td>
<td>101.1</td>
</tr>
<tr>
<td>average ( Z_S ) value</td>
<td>7.32</td>
<td>96.2</td>
</tr>
</tbody>
</table>

11.5.3 Oort cloud of comets

As we have already presented in sections 11.5.1 and 11.5.2, the found values of \( Z_S \) lead to various values of maximal distance between the Sun and the galactic equatorial plane. The result suggests that \( Z_S \) may play an important role in orbital evolution of comets of the Oort cloud. As a consequence, our knowledge of the mass of the Oort cloud may be incorrect.

The standard model of the Oort cloud considers that the Sun is situated in the galactic equatorial plane and \( Z_S = 0 \). Let us consider, as an example, that a comet is characterized by the following initial orbital elements: semi-major axis \( a_{in} = 5 \times 10^4 \text{ AU} \), eccentricity \( e_{in} \approx 0 \), inclination to galactic equatorial plane \( i_{in} = 90 \) degrees. The standard model yields 6 returns of the comet into the inner part of the Solar System during the existence of the Solar System (see Fig. 2 in Kómar et al. (2009)). Using the physical model presented by Klačka (2009) we obtain the following number of returns (see also Fig. 11.2):

- 9 for \( Z_S = 7.70 \text{ km/s} \),
- 12 for \( Z_S = 6.14 \text{ km/s} \).
Thus the mass of the Oort cloud of comets is 1.33-times or, even more times smaller when the greater number of returns occurs.

![Graph](image)

Figure 11.2: Time evolution of the comet eccentricity for various initial values of $\dot{z}(t = 0)$. Solid line for $\dot{z}(t = 0) = 7.70 \, km/s$ and dashed line for $\dot{z}(t = 0) = 6.14 \, km/s$

### 11.6 Discussion

The presented method of determining $\vec{V}_S$ by the least square method is based on the definition of a new general unit vector defined in Eq. (11.10). Taking into account various values of $\phi$ and $\theta$, we have obtained six independent systems of linear equation determining components of the vector $\vec{V}_S$. None of the systems, represented by Eqs. (A.0.1)-(A.0.6), contains the set...
of quantities \( \{ \dot{r}_i, \mu_\alpha i, \mu_\delta i \} \) simultaneously. The reason why only one, or maximally two of the components of the set \( \{ \dot{r}_i, \mu_\alpha i, \mu_\delta i \} \) are present in each of the final systems can be explained by the usage of the least square method, when the exponent in Eq. (11.14) equals to 2. If the exponent would be equal to 3, 4, ... also a set of equations with all three measured quantities \( \dot{r}_i, \mu_\alpha i, \mu_\delta i \) would exist.

(Mihalas and McRae-Routly 1968, p. 97) state that "Finally, if we add equations (5-10) to (5-13) and (5-11) to (5-14) to utilize the proper motion information as fully as possible, we obtain three equations of their form ..." (equations for \( X_S, Y_S, Z_S \)). This access of the authors yield result not consistent with our system A.0.2. It seems to us that is not possible to combine various equations in an arbitrary manner. Our access represented by Eqs. (11.9) - (11.16) yields combination of the measured quantities \( \dot{r}_i, \mu_\alpha i, \mu_\delta i \) in a unique way.

As Tables 11.1 - 11.3 show, the systems of equations A.0.2, A.0.4 do not yield results consistent with the direct method represented by Eqs. (11.7) - (11.8). The result for the direct method are summarized in Table 11.4. On the basis of Table 11.4 we obtain the following results:

\[
\begin{align*}
Z_S & = (7.32 \pm 0.15) \text{ km/s}^{-1}, \\
V_S & = (21.04 \pm 0.26) \text{ km/s}^{-1}.
\end{align*}
\]

(11.22)

Our values correspond to the standard solar motion. The value of \( Z_S \) is consistent with the results published by other authors (see values in Table 11.7). However, our value of \( V_S \) is greater than the values of other authors presented in Table 11.7. Especially the value presented by Binney and Merrifield (1998) and Dehnen and Binney (1998) is very small.

<table>
<thead>
<tr>
<th>source:</th>
<th>SM1</th>
<th>SM2</th>
<th>SM3</th>
<th>SM4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_S ) [km/s]</td>
<td>7.3</td>
<td>6.0</td>
<td>7.17 ( \pm ) 0.09</td>
<td>7.2 ( \pm ) 0.4</td>
</tr>
<tr>
<td>(</td>
<td>V_S</td>
<td>) [km/s]</td>
<td>19.5</td>
<td>15.4</td>
</tr>
</tbody>
</table>


11.7 Conclusions

We have calculated the solar motion with respect to three different sets of stars using the direct calculation and the approximative Least square method. We can see in subsections 11.4.1, 11.4.2 and 11.4.3 that the solar motion depends on the number of stars in our sample and their heliocentric distances. Our results correspond to the standard solar motion.

Our Least square method offered six independent systems of linear equations for the components $X_S$, $Y_S$ and $Z_S$ of the solar motion. The systems containing two of the three velocity components of the stars yield results for $X_S$, $Y_S$ and $Z_S$ not consistent with the values found by the direct method. This may be caused by used approximation (11.16). However, the same approximation yields acceptable results for the cases when only one of the velocity components of the stars (either radial velocity, or proper motion in right ascension, or proper motion in declination) is used. The value of $Z_S$ which we have found from the direct calculation fulfills the condition $Z_S \in (6.14, 7.70)$ km s$^{-1}$. The values are consistent with the values presented by other authors. However, our value of $V_S$ is larger than the presented by other authors.

The nonzero values of $Z_S$ are important in determining the maximal distances of the Sun from the galactic equatorial plane. The results are summarized in Tables 11.5 and 11.6. Moreover, the values of $Z_S$ are relevant in long-term evolution of comets of the Oort cloud. Using the newest form of the galactic tides, the orbital evolution of cometary orbital elements strongly depends on the value of $Z_S$. The cometary orbital evolution significantly differs from the results obtained by standard approach assuming a fixed Sun’s position in the galactic equatorial plane.

We have chosen only stars with the best quality index (A) in parallax, proper motions and radial velocities (SIMBAD Astronomical Database 2010), that minimized the influences of the measurement errors. As we can see in Table 11.1, our results from the direct calculation are in agreement with the standard solar motion 11.7. We want to know the solar motion regardless of the stellar type and hence we did not separate stellar types in our calculations.

The knowledge of the solar motion allows to simulate the solar motion with respect to galactic center. Especially $Z_S$, which represents the solar motion in direction perpendicular to the galactic equator, is used as a starting condition in simulation of the solar oscillations.

The uncertainty in the value of $Z_S$, $Z_S \in (6.14km/s, 7.70km/s)$, leads to the uncertainty in the mass of the Oort cloud of comets. A decrease in $Z_S$ in 20% leads to the decrease of mass of the Oort cloud in more than 30%.
Conclusion

This diploma thesis is divided into two parts.

The first and main part deals with the galactic halo models. We presented the actual models of galaxies and galactic halos. We calculated some physical quantities for each of the models. Especially values of the Oort constants and the shape of the rotation curve have been important, because they were compared with the observational data.

We have focused on the models of the galactic halos presented by Klačka (2009), Donato et al. (2009) and Gentile et al. (2009). The model presented by Klačka (2009) is based on the observations of our Galaxy. The model produces the flat rotation curve and the Oort constants which are in accordance with observations of our Galaxy. The conventional models presented by (Begeman et al. 1991, Burkert 1995, Navarro et al. 1996) describe other galaxies, but are not good to be applied to our Galaxy. Donato et al. (2009) and Gentile et al. (2009) present invariable quantities of the galaxies ($\rho_0 r_0$ and $\langle \Sigma_{0,\text{dark}} \rangle$). Thus we combined these various approaches to create more realistic model of the galactic halo. A final model gives the realistic values of the Oort constants, produces the flat rotation curve and also the values of $\rho_0 r_0$ and $\langle \Sigma_{0,\text{dark}} \rangle$ are in accordance with Donato et al. (2009) and Gentile et al. (2009), respectively.

A motivation for our future work is to create more realistic model of the galactic halos and to apply it to a problematics of the dark matter.

The second part of the diploma thesis discusses a solar motion. We derived a new method of the solar motion’s calculations. By using various methods we calculated a Sun velocity vector in respect of the local standard of rest. Finally, some applications were pointed out.
Resumé

Diplomová práca sa dá rozdeliť do dvoch častí. Prvá je zameraná na modely galaxií a galaktických hál. Druhá časť sa zaobere slnečným pohybom.

V súčasnosti všeobecne prijímanou skutočnosťou je existencia tmavej hmoty vo vesmíre. Takmer všetky kozmologické teórie predpokladajú jej existenciu, avšak žiadna z nich dostatočne nevyšetruje je podstatu ani zloženie. Sme schopní pozorovať len nepriame pôsobenia tmavej hmoty na svietiacu hmotu. Popis správania sa objektov v galaxiách sa stáva základom pre ďalšie štúdium tmavej hmoty vo vesmíre. Toto bol hlavný motivačný faktor pre napísanie tejto diplomovej práce.

V prvých dvoch kapitolách sme popísali súčasné predstavy a modely našej Galaxie i ostatných galaxií. Model prezentovaný Klačka (2009) sa stal základom pre vytvorenie nového realistieckejšieho modelu galaktického hala. Tento model je založený na pozorovaní našej Galaxie. Dáva hodnoty Oortových konštánt, ktoré sú v súlade s pozorovaniami a produkuje ploché rotačné krivky. Tento model je popísaný v podkapitole 1.2. Dôležité je uvedomiť si, že tento model obsahuje tri volné parametre α, aH, b, ktoré môžeme získať pomocou rôznych fyzikálnych podmienok.


V kapitole 3. sme získali vzťah pre strednú povrchovú hustotu galaktického hala a jej závislosť od parametra α v modeli Klačka (2009) sme znázornili na obr. Fig. 3.1. V nasledujúcich kapitolách 4. a 5. sme vypočítali hustotu hmotnosti, rotačnú rýchlosť a strednú povrchovú hustotu pre jednoduchý model (α = 1) resp. pre model Klačka (2009) kde (α = 0.174).
V kapitolách 6. a 7. sme sa zaoberali výpočtom fyzikálnych veličín \( r_0 \rho_0 \) resp. \( \langle \Sigma_{0,dark} \rangle \) pre jednoduchý model \((\alpha = 1)\) resp. pre model Kláčka (2009) kde \((\alpha = 0.174)\). Vypočítali sme aj hodnotu \( \alpha \) parametra, pre ktorú dáva model \((2.1)\) hodnoty \( r_0 \rho_0 \) resp. \( \langle \Sigma_{0,dark} \rangle \) zhodné s práčami Donato et al. (2009) a Gentile et al. (2009). Výsledná hodnota je daná vzťahmi \((6.8)\) resp. \((7.16)\). V nasledujúcej kapitole 8. sú hodnoty parametru \( \alpha \) získané rôznym spôsobom spresnené pomocou metód najmenších štvorcov a výsledné hodnoty sú znázornené v rovnici \((8.11)\).

V kapitole 9. sme vypočítali hodnoty Oortových konštánt v oblasti Slnka pre jednotlivé modely. Ako vidíme z porovnania Oortových konštánt vypočítaných z modelu Kláčka (2009) s parametrom \( \alpha \) získaného v kapitole 8.\((9.10)\) s reálnou hodnotou Oortových konštant \((1.7)\) zistíme, že nový model nedáva celkom reálne hodnoty.

V kapitole 10. sme vytvorili konečný model galaktického hala. Využili sme metódu najmenších štvorcov a určili sme optimálne hodnoty parametov \( \alpha, a_H, b \) \((10.6)\) vystupujúcich v \((2.1)\). Nový model dáva ploché rotačné krivky, Oortove konštanty v zhode s pozorovaniami a aj hodnoty veličín \( r_0 \rho_0 \) a \( \langle \Sigma_{0,dark} \rangle \) v zhode s práčami Donato et al. (2009) a Gentile et al. (2009). Teda splňa všetky, na začiatku stanovené, ciele. Vypočítali sme aj jednotlivé charakteristiky tohto výsledného modelu.

Ako motivácia do ďalšej práce je zostavenie ešte realistickejšieho modelu galaktického hala a následne ho aplikovať na problematiku tmavej hmoty vo vesmíre.

V poslednej 11. kapitole sa venujeme pohybu Slnka vzhľadom na miestny štandard pokoja. Je vypracovaná nová metóda určovania rýchlosti slnečného pohybu, porovnaná je s inými metódami a sú prezentované niektoré aplikácie slnečného pohybu. Z výsledkov je zrejmé, že pohyb Slnka v kolnom smere na rovini galaktického rovníka a rôzne hodnoty rýchlosti tohto pohybu majú výrazný vplyv na Oortov oblak, pričom sa v súčasnosti bežne tento pohyb zanedbáva.
Bibliography


Appendix A

Equations deduced from the Least square method

For arbitrary choice of $\phi$ and $\theta$, we get

A.0.1 Coefficients of $\sin^2(\theta) \cos^2(\phi)$

\begin{align*}
4.74 \sum_{i=1}^{N} r_i \mu_{\alpha_i} \cos(\delta_i) \sin(\alpha_i) &= X_S \sum_{i=1}^{N} \sin^2(\alpha_i) - Y_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\alpha_i), \\
4.74 \sum_{i=1}^{N} r_i \mu_{\alpha_i} \cos(\delta_i) \cos(\alpha_i) &= X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) - Y_S \sum_{i=1}^{N} \cos^2(\alpha_i). 
\end{align*} 

(A.1)
A.0.2 Coefficients of $\sin^2(\theta) \sin(\phi) \cos(\phi)$

\[
4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \cos(\alpha_i) \sin(\delta_i) + 4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \sin(\alpha_i) = 2X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \sin(\delta_i)
+ Y_S \sum_{i=1}^{N} (\sin^2(\alpha_i) - \cos^2(\alpha_i)) \sin(\delta_i)
- Z_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\delta_i),
\]

\[
-4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \sin(\alpha_i) \sin(\delta_i) + 4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \cos(\alpha_i) = -X_S \sum_{i=1}^{N} \sin^2(\alpha_i) \cos(\alpha_i) \sin(\delta_i)
+ 2Y_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \sin(\delta_i)
- Z_S \sum_{i=1}^{N} \cos(\alpha_i) \cos(\delta_i),
\]

\[
4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \cos(\delta_i) = X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\delta_i)
- Y_S \sum_{i=1}^{N} \cos(\alpha_i) \cos(\delta_i).
\]

(A.2)
A.0.3 Coefficients of $\sin(\theta) \cos(\theta) \sin(\phi)$

\[ - \sum_{i=1}^{N} \dot{r}_i \cos(\alpha_i) \sin(\delta_i) + 4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \cos(\alpha_i) \cos(\delta_i) = 2X_S \sum_{i=1}^{N} \cos^2(\alpha_i) \sin(\delta_i) \cos(\delta_i) \]
\[ + 2Y_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \sin(\delta_i) \cos(\delta_i) \]
\[ + Z_S \sum_{i=1}^{N} (\sin^2(\delta_i) - \cos^2(\delta_i)) \cos(\alpha_i), \]

\[ - \sum_{i=1}^{N} \dot{r}_i \sin(\alpha_i) \sin(\delta_i) + 4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \sin(\alpha_i) \cos(\delta_i) = 2X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \sin(\delta_i) \cos(\delta_i) \]
\[ + 2Y_S \sum_{i=1}^{N} \sin^2(\alpha_i) \sin(\delta_i) \cos(\delta_i) \]
\[ + Z_S \sum_{i=1}^{N} (\sin^2(\delta_i) - \cos^2(\delta_i)) \sin(\alpha_i), \]

\[ \sum_{i=1}^{N} \dot{r}_i \cos(\delta_i) + 4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \sin(\delta_i) = -X_S \sum_{i=1}^{N} (\cos(\delta_i) - \sin^2(\delta_i)) \cos(\alpha_i) \]
\[ - Y_S \sum_{i=1}^{N} (\cos^2(\delta_i) - \sin^2(\delta_i)) \sin(\alpha_i) \]
\[ - 2Z_S \sum_{i=1}^{N} \sin(\delta_i) \cos(\alpha_i). \]  

(A.3)
A.0.4 Coefficients of $\sin(\theta)\cos(\theta)\cos(\phi)$

\[-\sum_{i=1}^{N} \dot{r}_i \sin(\alpha_i) + 4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \cos(\alpha_i) \cos(\delta_i) = 2X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \cos(\delta_i)
\]
\[+ Y_S \sum_{i=1}^{N} (\sin^2(\alpha_i) - \cos^2(\alpha_i)) \cos(\delta_i)
\]
\[+ Z_S \sum_{i=1}^{N} \sin(\alpha_i) \sin(\delta_i),
\]

\[\sum_{i=1}^{N} \dot{r}_i \cos(\alpha_i) + 4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \sin(\alpha_i) \cos(\delta_i) = X_S \sum_{i=1}^{N} (\sin^2(\alpha_i) - \cos^2(\alpha_i)) \cos(\delta_i)
\]
\[\quad - 2Y_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \cos(\delta_i)
\]
\[\quad - Z_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\delta_i),
\]

\[4.74 \sum_{i=1}^{N} r_i (\mu_{\alpha,i} \cos(\delta_i)) \sin(\delta_i) = X_S \sum_{i=1}^{N} \sin(\alpha_i) \sin(\delta_i)
\]
\[\quad - Y_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\delta_i).
\]

(A.4)
A.0.5 Coefficients of $\sin^2(\theta) \sin^2(\phi)$

\[
4.74 \sum_{i=1}^{N} r_i \mu_{\delta,i} \cos(\alpha_i) \sin(\delta_i) = X_S \sum_{i=1}^{N} \cos^2(\alpha_i) \sin^2(\delta_i)
+ Y_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \sin^2(\delta_i)
- Z_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\delta_i) \cos(\delta_i),
\]

(A.5)
A.0.6 Coefficients of $\cos^2(\theta)$

\[\begin{align*}
\sum_{i=1}^{N} \dot{r}_i \cos(\alpha_i) \cos(\delta_i) &= -X_S \sum_{i=1}^{N} \cos^2(\alpha_i) \cos^2(\delta_i) \\
&\quad - Y_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \cos^2(\delta_i) \\
&\quad - Z_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\delta_i) \cos(\delta_i), \\
\sum_{i=1}^{N} \dot{r}_i \sin(\alpha_i) \cos(\delta_i) &= -X_S \sum_{i=1}^{N} \sin(\alpha_i) \cos(\alpha_i) \cos^2(\delta_i) \\
&\quad - Y_S \sum_{i=1}^{N} \sin^2(\alpha_i) \cos^2(\delta_i) \\
&\quad - Z_S \sum_{i=1}^{N} \sin(\alpha_i) \sin(\delta_i) \cos(\delta_i), \\
\sum_{i=1}^{N} \dot{r}_i \sin(\delta_i) &= -X_S \sum_{i=1}^{N} \cos(\alpha_i) \sin(\delta_i) \cos(\delta_i) \\
&\quad - Y_S \sum_{i=1}^{N} \sin(\alpha_i) \sin(\delta_i) \cos(\delta_i) \\
&\quad - Z_S \sum_{i=1}^{N} \sin^2(\delta_i). \quad \text{(A.6)}
\end{align*}\]