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Cosmological perturbations in a universe with an elastic component

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Introduction

In the most of the basic courses in cosmology, the universe is treated as isotropic and homogeneous, described by Robertson-Walker metric. This is just an approximation of a more interesting universe, with small departures from homogeneity and isotropy. Structures formed in the universe and anisotropies of the cosmic microwave background (CMB) are the main motivation for the study of these perturbations.

We will focus on the theory of perturbations produced throughout inflation, the exponential expansion of the early universe shortly after the Big Bang. According to the theory, quantum fluctuations are stretched during the inflationary era until their wavelength gets outside the Hubble horizon. Approximately at that moment, the perturbations start to behave as classical, which leads to the formation of structures and leaves an imprint on the cosmic microwave background as well.

Anisotropies of CMB are a very important tool distinguishing between different scenarios of inflation, because each scenario predicts different spectrum of inhomogeneities. Recently, the space observatory Planck has collected more precise data in comparison with those obtained from WMAP. These new data should provide enough information to eliminate at least some of existing models of inflation. One of the measurements with such potential is the observation of polarization of CMB [4].

Observation of the so called B-mode polarization would provide informations about tensor modes (gravitational waves) produced during inflation. The detection of the B-mode would be unfavourable to particle physics-motivated models, as primordial gravitational waves in such models are many orders smaller than observable values [11].[1]

This is the motivation for searching for a process which could enhance the amplitude of gravitational waves, or tensor perturbations. We propose a mechanism according to which cosmological perturbations are modified by interaction with an elastic component

[1] Recently in [1] it was announced that BICEP2 experiment had measured B-mode polarizations, but this result seems to be in a contradiction with previous limits from the Planck data. Joint analysis with the Planck data [5] disfavours that the measured polarizations were caused by primordial gravitational waves instead of cosmic dust.

of the universe. The motivation for such study is that a network of cosmic strings or domain walls with elastic properties can be used to simulate the effect of dark energy [6].

To describe the influence of the elastic continuum on the metric perturbations it is necessary to construct the theory of elasticity in the setting of general relativity (also called relasticity) [15]. It is a theory developed to describe diametrically different phenomena, mainly solid crusts of neutron stars. However, we use it to show that it is possible to modify cosmological perturbations for a proper choice of its parameters.

1 Standard scenario

1.1 Inflation

Cosmological inflation or just inflation is the exponential expansion of the early universe after the Big Bang. The idea of inflation was proposed in 1981 by Alan Guth [10]. Guth noticed that scalar fields could get caught in a local minimum of the potential. The energy of empty space would then remain constant for a while as the universe expanded. It would produce a constant rate of expansion, meaning that scale parameter would grow exponentially.

The scalar field responsible for inflation is called the inflaton. In the present one-inflaton scenarios, the inflaton field rolls down the potential until it gets close to its global minimum. The energy of the inflaton field is then transferred to the radiation and the evolution of the universe continues as in the Friedmann model.

Guth and others soon realized that the existence of an era of exponential expansion would solve some unsatisfactory aspects of the Hot Big Bang theory. The most important ones are the monopole problem, the flatness problem and the horizon problem.

1.2 Cosmological perturbations

Matter distribution in the universe can be considered homogeneous and isotropic on scales larger than a few hundred Mpc. This is consistent with the temperature fluctuations of the microwave background radiation, which show that the universe was extremely homogeneous and isotropic (with the accuracy of order 10^{-5} [16]) on all scales at the time of last scattering. On the other hand, a large scale structure is observed at the present time in the observable universe. Therefore, primordial inhomogeneities, serving as the seeds for structure formation, are an important part of the present cosmological models.

In the analysis of perturbations, we assume that the departure from homogeneity and isotropy is small, so that it can be treated within linearized theory. Because of the approximate homogeneity and isotropy of the universe and also negligible curvature at present, we take Robertson-Walker metric with zero curvature as the unperturbed metric.

Gauge-invariant variables. In a homogeneous and isotropic universe, the coordinates are chosen by the symmetry properties of the background, but there is no obvious choice of coordinates in the case of perturbed background. This means that by perturbing the coordinates, fictitious perturbations can be produced or physical perturbations can be removed. To distinguish physical from fictitious modes, it is helpful to find gauge-invariant variables, i.e. variables that do not depend on the choice of coordinates, such variables were first introduced by Bardeen in [2].

The perturbative part of the metric tensor can be decoupled into three independent modes of perturbations – scalar, vector and tensor perturbations. In [12], two gauge-invariant quantities describing scalar perturbations Φ, Ψ are introduced. If both Φ and Ψ are equal to zero, metric perturbations are fictitious and can be removed by a transformation of coordinates. For vector perturbations only one gauge-invariant quantity V_i (which satisfies $V_{i,i} = 0$) can be found. Two polarizations of the gravitational waves are tensor perturbations h_{ij}^{TT}.

Equations of motion. The dynamics of cosmological perturbations is described by Einstein equations. Einstein tensor can be split into the background term and the linear perturbation term, and the stress-energy tensor can be written in the same way. This leaves us with two sets of equations. One set describes the homogeneous and isotropic background and is known as the Friedmann equations [17] (they govern the evolution of scaling parameter of the universe a and its energy density ρ), and the other set describes the dynamics of the perturbations. Bot sets can be found, for example, in [12]. In standard scenario in which no anisotropic stress terms are considered, one of the predictions of the theory is that the variables Φ and Ψ are equal.

Origin of primordial perturbations Before the idea of inflationary cosmology appeared, observational data were explained by choosing the appropriate initial conditions for the universe. On the other hand, inflationary cosmology explains the origin of primordial inhomogeneities and predicts their spectrum, so that the testing of the theory by comparing its predictions with observations became possible after it has appeared.

During inflation, classical physics predicts that the inflaton field becomes homogeneous and isotropic throughout the inflating region. However, as a consequence of the quantum theory one obtains vacuum fluctuations of both scalar field and metric. These quantum fluctuations interact with inflaton field and therefore the inflationary parameters (inflationary potential) leave its imprint in the scalar perturbations.

During the inflationary era the curvature scale (Hubble radius) is approximately fixed at a constant value. On the other hand, the physical scale of perturbations grows, therefore the perturbations eventually get larger than the Hubble radius (the event horizon). The scalar and tensor perturbations are approximately constant when they are above the horizon (results from the linearized Einstein equations) and they can be stretched to very large scales. After the end of the inflation the perturbations, already as classical quantities, can once again get under the horizon, this time the particle horizon in a Friedmann universe.

Spectrum. Let us consider a generic perturbation Δ. To describe the sample of perturbations observable at the present moment from the Earth one treats this perturbation as a random field. So, we have an ensemble of possible perturbations as realizations of a random process. In simple inflationary models the perturbations generated by the vacuum fluctuations are Gaussian [11] and independent on the direction of the wavevector. Then the mean value of the square of the perturbation can be written in terms of the power spectrum $\mathcal{P}_\Delta$ as $\langle\Delta^2\rangle = \int \mathcal{P}_\Delta(k)dk/k$, where the integration runs over the wavenumber k (absolute value of the wavevector).

The inflationary models predict the dependence of the magnitude of the perturbations on its wavenumber. The power spectra are predicted to be approximately flat, while its magnitude for the scalar perturbations is given by the inflationary parameters.

1.3 Cosmic microwave background anisotropies

Primordial perturbations originated from quantum fluctuations. During the inflationary era, fluctuations were stretched to large scales with nearly unchanged amplitudes, and later, at the time of last scattering, they left their imprint on the cosmic microwave background that can be observed now. Thus, by observing the anisotropies of CMB the theoretical predictions about the creation and evolution of cosmological perturbations can be tested.

The CMB radiation is described to a high accuracy by the distribution of blackbody temperature T (after removing dipole component existing due to the movement of the observer with respect to the comoving coordinates) with small anisotropies δT originating from the primordial perturbations.

The effect of cosmological perturbations on temperature anisotropies can be calculated from collisionless Boltzmann equation [11]. For radiation propagating along the null geodesics observed from direction $\vec{N}$, the effect is given by

$$\frac{d}{dt}\left(\frac{\delta T}{\overline{T}}+\Phi\right)=\frac{\partial}{\partial t}(\Phi+\Psi), \qquad \frac{d}{dt}\frac{\delta T}{\overline{T}}=-\frac{1}{2}\frac{\partial h_{ij}^{TT}}{\partial t}N^i N^j. \tag{1.1}$$

First equation describes the effect of scalar perturbations called Sachs-Wolf effect and the second equation describes the effect of tensor perturbations.

Correlation function. To describe the anisotropies of CMB it is convenient to expand fractional perturbation of temperature $\delta T/T$ into spherical harmonics with a_{lm}^T being the expansion coefficients.

An important quantity describing CMB temperature fluctuations is the average of their product over the positions of the observer,

$$\left\langle \frac{\delta T}{\overline{T}}(\vec{N}_1)\frac{\delta T}{\overline{T}}(\vec{N}_2)\right\rangle=\sum_l \frac{2l+1}{4\pi}C_l^{TT}P_l(\vec{N}_1.\vec{N}_2), \tag{1.2}$$

where P_l denotes Legendre polynomial and $C_l^{TT}=\langle\left|a_{lm}^T\right|^2\rangle$.

In order to compare measurements with theory, the last step remains to be taken. It is the computation of average, where the temperature fluctuations are given in terms of the primordial perturbations by (1.1).

The effect of scalar perturbations is

$$C_{l(scalar)}^{TT}=\frac{2}{\pi}\int_0^\infty \left\langle\left|\left(\Phi_k+\frac{\delta_k^\gamma}{4}\right)j_l(k\chi_{rec})-\frac{3}{4k}\delta_k^{\gamma\prime}\frac{\partial j_l(k\chi_{rec})}{\partial(k\chi_{rec})}\right|^2_{t_{rec}}\right\rangle k^2dk, \tag{1.3}$$

where j_l is spherical Bessel function of order l, δ_k^γ is density contrast of the radiation to be derived from linearized Einstein equations and χ_{rec} is comoving distance of the surface of last scattering where the observed radiation separated from the matter at the moment t_{rec}. The average is given by the power spectrum of the primordial perturbations.

The effect of tensor perturbations is an integral effect along the null geodetics of the radiation,

$$C_{l(tensor)}^{TT}=\frac{1}{2\pi}\frac{(l+2)!}{(l-2)!}\int_0^\infty\left|\int_0^{k\chi_{rec}}\frac{\partial h_k^{TT}}{\partial x}\frac{j_l(x)}{x^2}dx\right|^2 k^2dk. \tag{1.4}$$

1.4 Polarization of the CMB

Polarization of the CMB is an observable consequence of the presence of cosmological perturbations in the early universe. It gives us a unique insight into the properties of relic gravitational waves, otherwise unobservable due to their small amplitude at present.

Polarization tensor P_{ab} is symmetric and traceless tensor quadratic in the electric field. This tensor is a suitable object describing the polarization of radiation.

Polarization tensor has two independent components, and as a result, one can represent it by two scalar functions, called E- and B-modes of polarization. It is convenient to do so, because B-mode is produced by tensor perturbations only. (There are some other processes not related to primordial perturbations which produce B-mode, however these can be separated from the measured data.)

Thomson scattering. Thomson scattering is the scattering of electromagnetic wave by a free charged particle in classical electrodynamics. After the scattering the wave becomes linearly polarized. This polarization depends on the direction of the incoming and observed wave. The contributions of incoming waves from different directions are additive, so if an unpolarized isotropic radiation is scattered on the particle then the observed radiation is unpolarized as well. However, if we consider an incoherent radiation coming to the charged particle from all directions with an anisotropic directional intensity, then the scattered wave may be polarized.

Recombination. Now let us have a look at the recombination. As already mentioned, before recombination the radiation is thermalized because of its frequent scattering on charged particles. The scattering is negligible after recombination. If the recombination was instantaneous, the originally unpolarized isotropic radiation would stay unpolarized and we would observe no polarization of the cosmic microwave background.

To obtain a nonzero polarization we have to extend the simplified model of recombination and suppose that the recombination is delayed; that is, the separation of radiation from matter lasts a finite time. During the delayed recombination the radiation develops a small anisotropy which manifests itself in the polarization of the radiation scattered at the end of the recombination. Visibility function $\Pi(t_1, t_2)$ is a quantity defined so that the probability that the wave observed at the time t_2 has been scattered for the last time at the time from interval $(t_1, t_1 + dt_1)$ is $\Pi(t_1, t_2)dt_1$. This quantity, which can be expressed in the terms of the optical depth throughout the recombination, enters calculation of the polarization tensor as it holds the information about the delay of recombination.

Correlation functions. To compare the observations to the theoretical model, it is convenient to define correlation functions analogically to the quantity (1.2), for example $\langle E(\vec{N}_1)\frac{\delta T}{T}(\vec{N}_2)\rangle$. The easiest way to obtain correlation coefficients C_l^{XY}, where X, Y assume values T, E, B, is to make an expansion of the polarization tensor into E-type and B-type tensor harmonics, which form a complete basis for second rank tensors on the sphere. We obtain coefficients a_{lm}^E and a_{lm}^B from the expansion which can be used to calculate coefficients $C_l^{XY} = \langle a_{lm}^X a_{lm}^{Y*}\rangle$. It is quite complicated to calculate the polarization tensor of the incoming radiation and then extract the coefficients and we do not show the formulas here. But some of the results are simple, $C_l^{BT} = 0$, $C_l^{BE} = 0$ and for scalar perturbations $C_l^{BB} = 0$ (scalar perturbations do not produce B-mode polarizations). It is noteworthy that $\langle XY\rangle$ (correlation function between anisotropy X and Y) is zero if X and Y are caused by different types of perturbation. The reason is that the mean value of any perturbation is zero, only the standard deviation of the perturbation is nonzero.

2 Extended model

In standard cosmological model, presented so far, the anisotropic stress is usually supposed to be zero. However, the anisotropic stress significantly modifies the evolution of cosmological perturbations. We study such effect in a model which includes an additional elastic continuum in the universe. The effect naturally depends on the parameters of the added continuum, therefore we need a theory describing elastic continuum in the frame of the theory of general relativity at first. Such theory is called relasticity (short for relativistic elasticity), with [7] being its first comprehensive study.

2.1 Relasticity

Einstein's theory of general relativity and mechanics of elastic continuum are two well known theories. Both have important applications, but in different fields of physics. General relativity usually assumes matter to be in the form of ideal fluid in its cosmological and astronomical applications. On the other hand, mechanics of elastic continuum is used in situations when Newtonian theory or gravitation is valid.

Elastic continuum. For simplicity, consider a homogeneous, isotropic and flat elastic continuum extending over all spacetime. In the nonrelativistic theory, the elastic properties of a homogeneous and isotropic continuum are determined by two Lame coefficients λ and μ, and when the continuum is deformed in a homogeneous and isotropic way, its stress tensor is given by one scalar quantity – the pressure σ. These quantities are defined analogically for a general relativistic homogeneous and isotropic continuum.

The parameters appear in the expression for the energy per particle ε in terms of the contravariant body metric tensor H^{AB} (push-forward of the contravariant spacetime metric tensor to the body space),

$$\varepsilon = \bar{\varepsilon} + \frac{1}{2}\sigma\delta H^A_A + \frac{1}{8}\lambda\left(\delta H^A_A\right)^2 + \frac{1}{4}\mu\delta H^A_B\delta H^B_A, \tag{2.1}$$

where tensor $\delta H_A{}^B$ is the deviation of the actual body metric tensor from the body metric tensor in the partially relaxed state (state with minimum energy per particle for given particle number density equal to $\bar{\varepsilon}$), with the first index lowered by the covariant body metric tensor in the partially relaxed state.

In a theory with our definition of λ and μ, the shear modulus of an elastic medium is $\mu_{shear} = \mu + \sigma$ (ideal fluids have $\mu_{shear} = 0$).

Equations of motion. The equations of the motion and their derivation from the Einstein equations can be found in [15] (the signature (-+++) and units in which $16\pi G = 1$ are used there).

The equations for homogeneous and isotropic background (Friedmann equations) remain unchanged,

$$\frac{\dot{a}^2}{a^2} = \frac{1}{6}\left(\rho_0 + a^{-3}\varepsilon\right), \qquad \frac{\partial \varepsilon}{\partial a} = -3a^{-1}\sigma, \tag{2.2}$$

However, the modification of the theory caused by additional elastic continuum manifests itself in the linearized equations at the perturbative level. The material constants ε, σ, λ and μ are not independent. In cosmology, when considered as functions of a, they are interconnected by relation

$$\frac{\partial \sigma}{\partial a} = -a^{-1}\left(2\sigma + 3\lambda + 2\mu\right). \tag{2.3}$$

Plane waves. In [15], the comoving proper time gauge is chosen to work in. Just one plane wave, for which the wave covector is chosen to be $(k, 0, 0)$ (rotational invariance of the background solution allows this without loss of generality of the solution), is introduced in the continuum. As a result, the perturbation to the background (Robertson-Walker) metric can be written as

$$\delta g_{00} = 0, \qquad \delta g_{0i} \equiv -ikh_{0i}(t)e^{ikx^1}, \qquad \delta g_{ij} \equiv a^2(t)h_{ij}(t)e^{ikx^1}, \tag{2.4}$$

where the signature is $(-,+,+,+)$. Tensor perturbations (gravitational waves) are given by h_{23} for $\times$ polarization and $(h_{22} - h_{33})/2$ for $+$ polarization.

Vector perturbations are described by the vectors $h_{1\alpha}$ and $h_{0\alpha}$, where $\alpha = 2, 3$ (two transversal directions). Later we will use $\tilde{h}_{1\alpha} \equiv -(a/\dot{a})h_{1\alpha}/4$ instead of $h_{1\alpha}$.

The comoving proper time gauge removes the ambiguity in the choice of the coordinates up to the remaining freedom of choosing the zero time hypersurface. Because of this freedom, the three remaining quantities h_{01}, h_{11} and h_{ii} (summed over i) describing scalar perturbations can be reduced to two shift-independent quantities y_{01} and y_{11}, defined by equations

$$h_{01} = y_{01} + y, \qquad h_{11} = y_{11} + 2\frac{\dot{a}}{a}y, \qquad h_{ii} = y_{11} + 6\frac{\dot{a}}{a}y, \tag{2.5}$$

the quantity y contains no additional physical information about scalar perturbations, since it is related to y_{01} by the equation $\dot{y} = -y_{01}(\varepsilon + \sigma)/(4\dot{a}a^2)$. In the calculations we will use the variable $\tilde{y}_{01} \equiv 6y_{01}\dot{a}/a$ instead of y_{01}.

The linearized equations for scalar (y_{01}, y_{11}), vector ($h_{0\alpha}$, $h_{1\alpha}$) and for tensor perturbations (h_{ij}^{TT}) can be found in [15].

2.2 Gauge invariant variables

The perturbations are defined in the specific gauge proposed in [15]. However, it is often more convenient to work with gauge invariant variables. Tensor perturbations are invariant by themselves, but $\tilde{y}_{01}$, y_{11}, $h_{0\alpha}$ and $\tilde{h}_{1\alpha}$ are gauge dependent, therefore we must find relations between them and the gauge invariant variables Φ, Ψ and V_α of [12]. In this way we replace rather unknown variables by well known gauge invariant variables suitable for further study of perturbations.

Comparing the two differently defined metric tensors, the invariant variables can be expressed in terms of the variables defined in the proper-time comoving gauge in the following way:

Scalar	$\Phi = -\left[\dot{y}_{01} + \dot{y} - \frac{a^2}{2k^2}\left(2\frac{\dot{a}}{a}\dot{y}_{11} + \ddot{y}_{11}\right)\right] e^{ikx^1}$
	$\Psi = \frac{\dot{a}}{a}\left(y_{01} - \frac{a^2}{2k^2}\dot{y}_{11}\right) e^{ikx^1}$
Vector	$V_i = i\frac{k}{a}\left(0, \quad h_{02} - \frac{a^2}{k^2}\dot{h}_{12}, \quad h_{03} - \frac{a^2}{k^2}\dot{h}_{13}\right) e^{ikx^1}$
Tensor	$h_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (h_{33} - h_{22})/2 & -h_{23} \\ 0 & -h_{23} & -(h_{33} - h_{22})/2 \end{pmatrix} e^{ikx^1}$

Table 2.1: Comparison of the gauge invariant perturbations and the perturbations in proper-time comoving gauge with wavevector $(k, 0, 0)$.

2.3 One-component universe

Let us suppose that the universe is filled only by a continuum satisfying the equation of state $\sigma = w\varepsilon$, where w is a constant dependent on the type of continuum (e.g. $w = 0$ for dust, $w = 1/3$ for radiation). Friedmann equations with this equation of state determine the evolution of energy density to be given by $\varepsilon \propto a^{-3w}$, and consecutively the evolution of scale parameter to be given by $a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}}$ (t is cosmological time and η is conformal time). Here we have constrained the equation-of-state parameter w to the values $w \in \mathbb{R}\backslash\{-1/3, -1\}$ (two excluded values are discussed later).

Let us define sound speeds which characterize sound waves in a solid and appear in the equations of motion for perturbations (see a hint of the derivation in [6]). Transversal and longitudinal sound speeds in a one component universe are

$$c_{s\parallel}^2 = w + \frac{4\xi}{3(1+w)}, \qquad c_{s\perp}^2 = \frac{\xi}{1+w}, \tag{2.6}$$

where the shear stress is parametrized by $\xi \equiv \mu_{shear}/\varepsilon$.

Analytical solutions. The equations governing cosmological perturbations are solvable only numerically in a general multicomponent universe, however analytical solutions do exist in a one-component universe in case the parameters w and ξ are both constant. These solutions are linear combinations of Bessel functions J_n and Y_n.

Let us define two constants entering the solutions

$$n_0 \equiv \frac{3(1-w)}{2(1+3w)}, \qquad n_\xi \equiv \sqrt{n_0^2 - 4c_{s\perp}^2\frac{6(1+w)}{(1+3w)^2}}. \tag{2.7}$$

and three variables: $\phi_{\xi\parallel} \equiv c_{s\parallel}k\eta$, $\phi_{\xi\perp} \equiv c_{s\perp}k\eta$ and $\phi \equiv k\eta$. Then the solutions are

$$\tilde{y}_{01} = \eta^{-n_0}\left[A_\xi^S J_{n_\xi}(\phi_{\xi\parallel}) + B_\xi^S Y_{n_\xi}(\phi_{\xi\parallel})\right], \tag{2.8}$$

$$h_{0\alpha} = \eta^{\frac{3}{2}}\left[A_\xi^V J_{n_\xi}(\phi_\perp^\xi) + B_\xi^V Y_{n_\xi}(\phi_\perp^\xi)\right], \tag{2.9}$$

$$h_{\vec{k}} = \eta^{-n_0}\left[A_\xi^T J_{n_\xi}(\phi) + B_\xi^T Y_{n_\xi}(\phi)\right], \tag{2.10}$$

and the remaining perturbations are easy to calculate if the respective sound speed is nonzero,

$$y_{11} = \frac{w}{c_{s\parallel}^2}\tilde{y}_{01} - \frac{1+3w}{6c_{s\parallel}^2}\eta\tilde{y}_{01}', \qquad \tilde{h}_{1\alpha} = \frac{1+3w}{8c_{s\perp}^2}\eta h_{0\alpha}' - \frac{3w}{4c_{s\perp}^2}h_{0\alpha}. \tag{2.11}$$

The coefficients A and B are constants dependent on the initial conditions of the perturbations.

Asymptotic solutions. Two asymptotic regimes can be observed for all three types of perturbations. The magnitude of the perturbation above the respective sound horizon (argument of Bessel functions is small) increases/decreases in conformal time by a power law and the solution enters oscillatory mode after the perturbation crosses the sound horizon (the argument is large).

The long-wavelength expansions of the perturbations (in fact expansion of the Bessel functions, which can be found, for example, in [8],[9]) yield

$$\tilde{y}_{01} \doteq \hat{\eta}^{-n_0} \left\{ \left[\cosh(n_\xi \ln \hat{\eta}) + \frac{3}{2} \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} \right] \tilde{y}_{01,init} - \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} \frac{6c_{s\parallel}^2 y_{11,init}}{1+3w} \right\}, \quad (2.12)$$

$$h_{0\alpha} \doteq \hat{\eta}^{\frac{3}{2}} \left\{ \left[\cosh(n_\xi \ln \hat{\eta}) - n_0 \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} \right] h_{0\alpha,init} + \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} 4c_{s\perp}^2 \tilde{h}_{1\alpha,init} \right\}, \quad (2.13)$$

$$h_{\vec{k}} \doteq \hat{\eta}^{-n_0} \left\{ \left[\cosh(n_\xi \ln \hat{\eta}) + n_0 \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} \right] h_{\vec{k},init} + \frac{\sinh(n_\xi \ln \hat{\eta})}{n_\xi} \eta_{init} h'_{\vec{k},init} \right\}, \quad (2.14)$$

where we imposed initial conditions at the moment η_{init} and we used notation $\hat{\eta} \equiv \eta/\eta_{init}$.

In standard scenario it is often emphasized that a perturbation above horizon cannot oscillate, because distant regions of the long-wavelength perturbation are not causally connected. However, as it turns out this is not true in the scenario with large enough shear stress when oscillatory mode exists even above the horizon.

The order of Bessel function n_ξ, which determines the behavior of the solutions, according to equation (2.7) is purely imaginary if $\xi > \xi_{crit} \equiv 3(1-w)^2/32$. Then the hyperbolic functions in the asymptotic expressions acquire imaginary arguments and become trigonometric functions of real arguments. As a result the perturbations described by these expressions oscillate.

If we want to work with the gauge invariant variables Φ and Ψ it is not sufficient to use leading order terms in $\tilde{y}_{01}$ and y_{11}, but higher order corrections are needed too. The reason for this is that the next-to leading terms get dominant in Φ and Ψ after sufficiently large period of time.

Special cases. So far we have omitted a few special values of the parameters in the parametric space of w and ξ. Le us analyze solutions describing perturbations for these values.

In former calculations we have excluded de Sitter universe ($w = -1$) and "curved" universe ($w = -1/3$). The solutions are qualitatively different from the solutions presented so far.

In de Sitter universe with nonzero shear stress we obtain $\tilde{y}_{01} = y_{11} = 0$ and $h_{0\alpha} = \tilde{h}_{1\alpha} = 0$. The general solution for tensor perturbations (2.10) stays valid for all values of ξ.

In a "curved" universe, the solutions are polynomials when expressed in cosmological time,

$$\tilde{y}_{01} = \tilde{A}^S_\xi t^{m_\parallel - 1} + \tilde{B}^S_\xi t^{-m_\parallel - 1}, \quad h_{0\alpha} = \tilde{A}^V_\xi t^{m_\perp} + \tilde{B}^V_\xi t^{-m_\perp}, \quad h_{\vec{k}} = \tilde{A}^T_\xi t^{m_\times - 1} + \tilde{B}^T_\xi t^{-m_\times - 1}, \quad (2.15)$$

where the coefficients $\tilde{A}$ and $\tilde{B}$ are constants of integration and the three parameters $\mathbf{m} \equiv (m_\parallel, m_\perp, m_\times)$ are defined in terms of the respective sound speeds $\mathbf{c} \equiv (c_{s\parallel}, c_{s\perp}, 1)$ as $m_p \equiv \left(1 - c_p^2 k^2 \mathcal{H}^{-2} - 6\xi\right)^{\frac{1}{2}}$.

Other special cases are those of zero sound speeds. If $c_{s\parallel} = 0$ then

$$\tilde{y}_{01} = \hat{\eta}^{\frac{6w}{1+3w}} \tilde{y}_{01,init}, \qquad y_{11} = \left[(1+3w)\frac{k^2\eta^2}{30} + \frac{1+w}{1+3w}\right] \tilde{y}_{01} + C_{11} \hat{\eta}^{-\frac{3(1+w)}{1+3w}}, \qquad (2.16)$$

and if $c_{s\perp} = 0$ then

$$h_{0\alpha} = \hat{\eta}^{\frac{6w}{1+3w}} h_{0\alpha,init}, \qquad \tilde{h}_{1\alpha} = \left[\frac{(1+3w)^2}{8(1-9w)} k^2\eta^2 + \frac{1+w}{1-w}\right] h_{0\alpha} + C_{1\alpha} \hat{\eta}^{\frac{3(1+w)}{1+3w}}, \qquad (2.17)$$

where C_{11} and $C_{1\alpha}$ are given by initial condition imposed on y_{11} or $\tilde{h}_{1\alpha}$ respectively.

Restrictions on parameters The solutions by themselves offer no direct restrictions on parameters. However, constraints on the parameters ξ and w can be obtained from physically motivated restrictions on the sound speeds. One constraint follows from the condition that the sound speeds in the continuum should be less than or equal to the speed of light. If we require the stability of perturbations (real sound speeds) we get another constraint. Put together, the conditions are $0 \leq c_{s\parallel}^2 \leq 1$ and $0 \leq c_{s\perp}^2 \leq 1$.

Figure 2.1 depicts regions in our parametric space with different behavior of perturbations. Regions satisfying restrictions on the longitudinal and transversal sound speed are hatched from upper left to lower right and from upper right to lower left respectively. The thin line in the figure depicts critical shear stress ξ_{crit}.

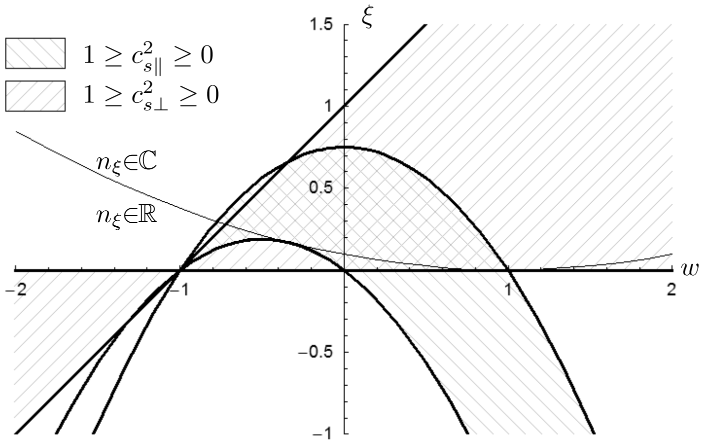

Figure 2.1: Regions in parametric space with restricted sound speeds.

2.4 Phase transition

Let us continue to study the perturbations in a one-component universe but we modify the underlying scenario. We will now suppose that the originally zero shear modulus assumes a nonzero value due to some phase transition in the continuum filling the universe, while the equation of state of the continuum remains the same with the previous value of the parameter w.

In the former scenario we have supposed that the ratio $\xi \equiv \mu_{shear}/\varepsilon$ is constant, therefore we did not consider terms proportional to $\partial_\eta \xi$ throughout the calculation. The modified scenario requires a time dependent ratio ξ, which rises from initially zero value to a nonzero constant ζ.

Matching conditions To be able to use the results obtained in a one-component scenario without shear stress, let us suppose that the phase transition took place instantaneously at the conformal time η_s, so that the shear stress is given by a step function. The motivation for this assumption is that the equations are then valid always except for the moment η_s, and all we need to do this is to match the solutions before and after that moment. The matching conditions are chosen so that the quantities $\tilde{y}_{01}$, y_{11}, $h_{0\alpha}$, $h_{1\alpha}$ and $h_{\vec{k}}$, $h'_{\vec{k}}$ are continuous at the moment η_s.

Some of the other variables are still discontinuous at the moment of phase transition. The reason for this behavior is that the jump in the shear stress parameter ξ produces a jump in the energy momentum tensor as well, which reflects itself in the metric via Einstein equations.

Long-wavelength perturbations. Here we restrict ourself to the analysis of perturbations which are above horizon at the time of phase transition and stay above horizon for a sufficiently long time after the transition as well, so that the decaying part of the solutions become negligible. If we require that the parameter ζ is small then we obtain quite simple formulas

$$\Phi_{\vec{k}} \doteq \left(\frac{\eta}{\eta_s}\right)^{-6\zeta}(1+4\zeta)\Phi_{\vec{k},init}, \tag{2.18}$$

$$V_{\alpha\vec{k}} \doteq -6\zeta\left(\frac{\eta}{\eta_s}\right)^{-6\zeta}\hat{\eta}^{-\frac{2}{1+3w}}\left[V_{\alpha\vec{k},init} - \frac{3(1+w)}{2(1+3w)}\frac{h_{1\alpha,init}}{ik\eta_{init}}\right], \tag{2.19}$$

$$h_{\vec{k}} \doteq \left(\frac{\eta}{\eta_s}\right)^{-6\zeta}(1+6\zeta)h_{\vec{k},init}. \tag{2.20}$$

Remark on linearity. The factors $(k\eta_s)^{-2}$ are present in the formula for the jump of the variable $\Phi_{\vec{k}}$. These factors lead to very large jumps in long-wavelength perturbations at the time of the phase transition.

Let us consider a physically motivated "toy" scenario which will give us an additional input. Suppose that the solid which forms after the phase transition is radiation-like (with equation of state $\sigma = \varepsilon/3$); then the universe can be effectively treated as one-component and we can use the solutions we have obtained earlier. At the time of recombination $\eta_{rec} > \eta_s$ (in our simplified scenario recombination occurs in the radiation dominated era) the imprint of scalar perturbations is left on the angular power spectrum of CMB, which we observe at present. The observations introduce a scale into the theory in the form of the approximately flat power spectrum for long-wavelength perturbations at the time of recombination, $\mathcal{P}_{\Phi,rec} \approx 2\times 10^{-9}$ [14].

With this scale in our model we find out that the perturbation Φ can increase sufficiently to invalidate the linearized theory. For a GUT energy scale of the inflation $\rho_{GUT} \approx (2\times 10^{16}GeV)^4$ and number of e-foldings throughout the inflation put equal to minimum admissible value, $\mathcal{N} \approx 62$ (see [18]), we obtain that the magnitude of the perturbation Φ right after the phase transition $\left\langle |\Phi_{s+}|^2\right\rangle \approx \zeta^2\rho_s/eV^4$, where ρ_s is the energy scale of the phase transition.

Therefore, for $\zeta > (\rho_s/eV)^{-2}$ the phase transition scenario collapses if one uses the gauge invariant variables. The proper-time comoving gauge variables are a better choice to analyze effects of the phase transition because they do not disqualify the perturbative approach at the moment η_s.

2.5 Cosmic microwave background

The observations of CMB anisotropies are an important input into the standard theory of cosmological perturbations. This remains valid in the modified theory as well. In fact, the modifications are minor compared to the amount of calculations needed in the standard scenario.

In addition to the modification of the evolution of the perturbations, several corrections in the formulas determining the anisotropies of CMB must be made, too. In the standard scenario without shear stress, the perturbations Φ and Ψ are identical, but after introducing shear stress the two functions evolve differently and therefore one needs to correct the formulas where equality $\Phi = \Psi$ is used. Furthermore, the density contrast of radiation must be calculated from

$$\delta^{\gamma} = 4\Psi - \frac{2}{3}y_{11}. \tag{2.21}$$

With these modifications the calculation of the temperature and polarization anisotropies of CMB is straightforward. (The complete formulas which should be used for correlation coefficients C_l^{XY} are not presented here.)

Summary

In this work we studied the formation, evolution and observation of cosmological perturbations. All of these aspects are discussed in first part, which is meant as a pedagogical introduction and theoretical background to the chapter Extended model.

First we introduced briefly the concept of cosmological inflation, after listing the three cosmological problems which are solved by it. A consequence of exponential expansion of the universe during the inflationary era is the formation of cosmological perturbations. We explained this process and the evolution of perturbations in general. Then we studied observational consequences of the presence of perturbations in our universe, the temperature and polarization anisotropies measured in CMB radiation.

We modified the standard scenario in the fifth chapter called Extended model. An isotropic solid continuum with elastic properties was introduced there and its effect on the evolution of the perturbations was studied. It turned out that the elastic properties of such continuum can be parametrized by one parameter, shear modulus. All calculations were made in the proper-time comoving gauge, but then they were "translated" to more commonly used gauge invariant variables first introduced by Bardeen.

The main focus of the work was on a one-component universe, with constant ratio of shear modulus to energy density. The solutions for cosmological perturbations in such universe were expressed in terms of Bessel functions, which offered us new possibilities in the analysis of the solutions. The standard solutions found in the literature were shown to be a special case of our solutions. Solutions which could not be expressed in terms of Bessel functions were treated separately and physically motivated restrictions were imposed on the parameters.

We presented asymptotic solutions with emphasis put on long-wavelength solutions. We found that perturbations above the horizon decrease and can even oscillate despite the arguments presented in the literature, provided the shear modulus of the continuum is sufficiently large.

Then we modified the scenario. We added a specific time dependence of the shear stress to the theory, assuming that the initially zero shear stress parameter instanta-

neously increased to a final value at the moment of a phase transition. This modification brings some peculiarities to the evolution of the cosmological perturbations, which we have discussed. The complete solutions were found by matching the solutions before and after the phase transition. This scenario is advantageous in that that it allows to modify the evolution of cosmological perturbations without the need to modify the theory of their formation, so that the calculations presented in the theoretical introduction stay valid.

We also included a very quick review of the models of elastic solid into the chapter "Extended model", and supplemented our work by several appendices, meant to ease the orientation of the reader in the text.

List of publications and citations

Current journals

- V. Balek, M. Škovran. *Effect of radiation-like solid on CMB anisotropies*, Class. Quant. Grav. 32:015015, 2015. [arXiv:gr-qc/1402.4434]

  cited by: [3],[13]

Preprint journals

- M. Škovran. *Analytical solutions for cosmological perturbations in a one-component universe with shear stress.* [arXiv:gr-qc/1501.07262]

  accepted for publication in Int. J. Mod. Phys. D

- V. Balek, M. Škovran. *Suppression of large-scale perturbations by stiff solid.* [arXiv:gr-qc/1501.07262]

  submitted to Phys. Rev. D

- V. Balek, M. Škovran. *Cosmological perturbations in the presence of a solid with positive pressure.* [arXiv:gr-qc/1401.7004]

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Other

- M. Škovran. *Cosmological Perturbations.*
  Talk at Theoretical Physics Workshop and Summer School, Svit, 2011

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  Poster on 9th Vienna Central European Seminar, 2012

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