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Autoreferát dizertačnej práce

Bezkolízna po častiach Bézierova cesta nízkeho stupňa medzi prekážkami z regulárnych kužeľosečiek

na získanie akademického titulu philosophiae doctor
v odbore doktorandského štúdia

9.1.7 Geometria a topológia

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Oponenti: ....
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Obhajoba dizertačnej práce sa koná .... o .... h
pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie ....

Geometria and topológia – 9.1.7 Geometria a topológia

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1 Introduction

Bézier curves play an important role in geometric modeling, Computer Aided Geometric Design (CAGD) and computer graphics systems due to their properties. Path planning (as a part of motion planning) is a major topic in robotics. It involves finding a collision-free strategy from the current location, or configuration, to a desired goal location, or configuration. It is a purely geometric process that is only concerned with finding a collision-free path regardless of the feasibility of the path. Theory of Minkowski space served as a mathematical equipment since it has many practical applications also in geometric modelling and CAGD.

We decide to study a mutual position of Bézier curve and regular quadric in three dimensional Euclidean space. The gained knowledge are applicable in path planning where a path is represented by the Bézier curve and the obstacles represented by quadrics. We start with quadratic curves and splines, later we extend the propositions for cubic curves because of their better properties. Another application can be found in searching for pointwise space-like curves in Minkowski space where the light-cone is regular quadric.

2 Theoretical background

Let $\mathbb{E}^3$ be three dimensional vector Euclidean space formed by vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ with scalar product $\langle \cdot\,, \cdot \rangle \colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$. Let $M_{3,3}(\mathbb{R})$ be the set of 3×3 matrices with real coefficients. A quadratic form is the map $q \colon \mathbb{E}^3 \to \mathbb{R}$, where $q(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{Q}\boldsymbol{x}^\top$ for the symmetric $\boldsymbol{Q} \in M_{3,3}(\mathbb{R})$.

By a standard construction, we get three dimensional affine Euclidean space $\mathbb{R}^3$ formed by points $X = [x_1, x_2, x_3]$. Let $M_{4,4}(\mathbb{R})$ be the set of 4×4 matrices with real coefficients. Let $\boldsymbol{Q_\kappa} \in M_{4,4}(\mathbb{R})$ be regular and symmetric. An image of a *regular quadric* κ is the set of points $\{[x_1, x_2, x_3] \in \mathbb{R}^3 \colon (x_1\ x_2\ x_3\ 1)\boldsymbol{Q_\kappa}(x_1\ x_2\ x_3\ 1)^\top = 0\}$. The matrices 4×4 and extended coordinates $(X, 1) = (x_1, x_2, x_3, 1)$ are used to express an arbitrary translation and rotation of the quadric by one matrix. Moreover, projective properties are formulated in a more natural way. The used properties of affine and projective quadrics are in [3]. The intersection of a quadric and a plane is a *conic section* K [5, 14].

Bézier curve segment of degree n in the space $\mathbb{R}^d, d \in \mathbb{N}, d \geq 2$ is a polynomial map $b\colon [0,1] \to \mathbb{R}^d$ given by $b(t) = \sum_{i=0}^{n} B_i^n(t)\, V_i$. The points $V_i \in \mathbb{R}^d$ are called *control points*, the functions $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ for $i \in \{0, \ldots, n\}$ are Bernstein polynomials of degree n. More about the properties of Bézier curves can be found e. g. in [8, 12, 17, 20].

The state space for motion planning is a set of all possible transformations that may be applied to the robot. This will be referred to as the configuration space $C = C_{free} \cup C_{obs}$ [16], where C_{free} is a free space and C_{obs} represents an obstacle space. There are two basic approaches how to find the collision-free path from the initial configuration to the goal configuration. The first approach is *combinatorial motion planning* [15], which means that from the input model the algorithms build a discrete representation that *exactly* represents the original problem. The second one is *sampling-based motion planning* [1, 7], which refers to algorithms that use collision detection methods to sample the configuration space and conduct discrete searches that utilize these samples. Some of these algorithms use Bézier curves [13, 21, 22, 23] or B-spline curves [18, 19].

3 Goals and tasks

We formulate three main goals. Let $\mathbb{R}^3$ be the three dimensional Euclidean space with obstacle represented by regular quadric κ. Let the point A be the start and the point B be the finish. We find all quadratic Bézier paths starting at A and ending at B representing collision-free path with respect to an obstacle κ.

The extension of the problem is the existence of more obstacles. Let $O = \{O_1, \ldots, O_m\}$ be the set of obstacles represented by regular quadrics. Let $P = \{P_1, \ldots, P_n\}$ be a given set of points such that $n \geq 2$, where P_1 is the start of the path and P_n is the end of the path. We want to find the conditions for C^0, resp. C^1 piecewise quadratic spline representing collision-free path with respect to obstacles O. Moreover, the spline have to start at P_1, pass through the points $P_2, \ldots, P_{n-1}$ and end at the point P_n.

The observations from the quadratic paths are applied on the collision-free paths represented by planar cubic Bézier curves. So, we want to formulate some theorems and hypothesis about collision-free cubic path with respect to an obstacle κ.

4 Quadratic collision-free Bézier path

Let us consider the Euclidean space $\mathbb{R}^3$, let $\langle O, e_1, e_2, e_3 \rangle$ be an affine coordinate system, with a regular quadric κ represented by the matrix $\boldsymbol{Q_\kappa}$. Let the points $A, B \in \mathbb{R}^3$ be fixed and $\boldsymbol{a} = A - O, \boldsymbol{b} = B - O$ are their position vectors. Assuming the quadric is an enclosing volume of some obstacle and the points A, B are start and end position of a robot we require $q(\boldsymbol{a}) > 0$ and $q(\boldsymbol{b}) > 0$, i.e. A, B lie outside the quadric κ. We look for all collision-free (relative to quadric) paths supply by quadratic Bézier curves from the point A to the point B. In other words, we look for the set of all such points C that the Bézier curve $b_{ACB}\,(t)$ lie out of quadric. Thus, for all points $X \in b_{ACB}\,(t)$ and their position vectors $\boldsymbol{x}$ the inequality $q(\boldsymbol{x}) > 0$ holds. In applications, $A \neq B$ yields, however we solved also the case $A = B$ for the sake of completeness.

A generic quadratic Bézier curve is a part of a parabola, so it lies in the affine plane $\rho \subset \mathbb{R}^3$. Since the given points $A, B \in \rho$, the construction of the plane ρ may have several degrees of freedom depending on their positions. The intersection of the quadric κ and the plane ρ is a conic section K (see fig. 1(a)). The figures 1(b–g) show all cases how the set S of all points X that $q(\boldsymbol{x}) > 0$ in the possible types of plane ρ looks like. The collision-free Bézier curve $b_{ACB}\,(t) \subset S$. We present a solution in the plane ρ for each type of conic section $K \neq \emptyset$ separately and the planar results can be put together to form the spatial result.

Let $\langle O, x, y \rangle_\rho$ be any Cartesian coordinate system in the plane ρ. Let $A = [a_x, a_y]$, $B = [b_x, b_y]$ and $C = [c_x, c_y]$ be the local affine coordinates of the control points in $\langle O, x, y \rangle_\rho$. Let $V_\rho(A, B)$ be a set of points $C \in \rho$ such that the curve b_{ACB} is collision-free. Then, we say that $V_\rho(A, B)$ is a *set of admissible solutions* in the plane ρ with respect to A, B. If no confusion arises, we say the set of admissible solutions. By $V_\rho^v(A, B)$, we denote the set of points $C \in \rho$ such that $b_{ACB} \cap K$ contains only the points of contact of order 2 between the Bézier curve and the conic section.

4.1 Set of admissible points of contact

We say that the set $D \subset K$ is the *set of points of contact* between K and the set of all b_{ACB} if for any point $X \in D$ there is a point C such that $C \in V_\rho^v(A, B)$ and $X \in b_{ACB} \cap K$.

We say that the curve b_{ACB} *touches* a connected component of the regular conic section K from *outside (inside)*, if their common tangent is (is not) separating. Then,

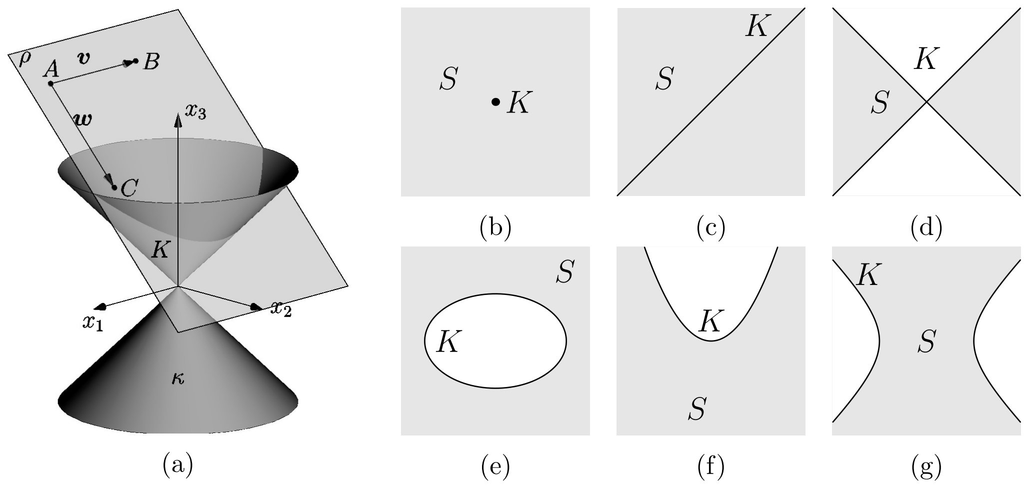

Figure 1: (a) Plane ρ spans points A, B, C. In case of their non-collinearity, they generate ρ as affine hull. The conic section K is the intersection of the quadric κ and the plane ρ. (b–g) Let $K \subset \rho$ be the conic section (point, double line, pair of lines, ellipse, parabola, hyperbola). The set S consists of all points lying out of quadric in the plane ρ.

the point of contact is called *exterior (interior) point of contact.* The set of all exterior (interior) points of contact is denoted D_{ext} (D_{in}).

We describe the set $D = D_{ext} \cup D_{in}$ for various regular types of the conic section K. The polar lines of the points A, B have the major influence on the shape of the set D. Let K be an ellipse. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of exterior and one arc of interior points of contact or one or two arcs of interior points of contact. Let K be a parabola. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of interior points of contact. Let K be a hyperbola. Then, the set of the points of contact D is either two arcs of the exterior points of contact or one arc of exterior and one arc of interior points of contact.

Now, let K be a singular conic section. In the case of $K = \{V_Q\}$, where V_Q is the top of the isotropic cone Q, we have $D = \{V_Q\}$. If $K = p$, where p is an isotropic double line, we must distinguish two cases. If A, B lie in the opposite half-planes generated by the line p, we have $D = \emptyset$. If A, B lie in the same half-plane, the set of points of contact $D = p$. The last singular case is $K = p \cup r$, where p, r are a pair of distinct isotropic lines. Then, there are two regions of points lying out of K in the plane ρ. If A, B lie in the different regions, there is no collision-free Bézier curve b_{ACB}. Let A, B lie in the same region, which is determined by two half-lines $\overrightarrow{V_Q P} \subset p$

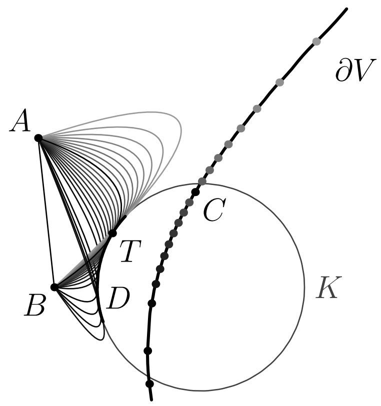

Figure 2: The boundary map σ maps the points of the arc D to the points on ∂V, see that $\sigma(T) = C$.

and $\overrightarrow{V_Q R} \subset r$. Let $S_p \in p$ and $S_r \in r$ are the points of contact of the Bézier curve b_{AC_uB}, i.e. $b_{AC_uB} \cap K = \{S_p, S_r\}$ is double contact. Then, $D = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$. The special case is $S_p = S_r = V_Q$.

4.2 Boundary map

For the given points A, B, and suitable $X \in K$ and the tangent line ℓ_X at X to K, the Bézier curve b_{ACB} touching the conic section K is clearly identified. In order to find the middle control vertex C, we use the *boundary map* σ such that $\sigma(X) = C$ with the form

$$\sigma(X) = \frac{b(t_0) - B_0^2(t_0)A - B_2^2(t_0)B}{B_1^2(t_0)},$$

where $t_0 \in [0, 1]$ is a solution of the equation

$$0 = \alpha t^2 + 2\beta t + \gamma$$

and for $A = [a_x, a_y, 1],\ B = [b_x, b_y, 1],\ X = [x_0, y_0, 1]$ are

$$\begin{aligned}
\alpha &= (A - B)\boldsymbol{Q_K} X^\top, \\
\beta &= -A\boldsymbol{Q_K} X^\top, \\
\gamma &= -\beta.
\end{aligned}$$

4.3 Set of admissible solutions

The set of all such points C that Bézier curve $b_{ACB} \cap K \subset D$ yields, is the boundary of the set of admissible solutions $V_\rho(A,B)$. We denote it $\partial V_\rho(A,B)$. We say that the boundary of the set of admissible solutions $\partial V_\rho(A,B)$ *is generated* by the set D mapped by the boundary map σ.

If the conic section $K = p$, then the boundary of the set of admissible solutions ∂V is the parallel line with the line p. If $K = p \cup r$, then the set of points of contact $D = \overrightarrow{S_pP} \cup \overrightarrow{S_rR}$ (in special case $S_p = S_r = V_Q$). The set ∂V consists from two half-lines parallel with p, resp. r, connected in the point C_u.

The boundary of admissible solutions $\partial V_\rho(A,B)$ consists of one or two continuous unbounded curves with degree at most four. These curves divide the plane ρ into few regions. Some of them are the components of set of acceptable solution $V_\rho(A,B)$. The position of the points A, B and the fact, whether the boundary was generated by D_{ext} of D_{in}, determine which regions belong to the set $V_\rho(A,B)$. For the two given points A, B and the conic K, the set of acceptable solutions $V_\rho(A,B)$ consists of one or two regions. It depends on the number of arcs in the set D and on the type of the conic section K.

5 Collision-free piecewise quadratic spline

In applications, a frequent task is to find a path from the starting point to the ending point while passing through the given set of points avoiding obstacles. Also, there may be more than one obstacle in the scene. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be the given set of points in $\mathbb{R}^2$. Let $\mathcal{O} = \{O_1, \ldots, O_m\}$ be the given set of obstacles represented by conic sections as their bounding objects. We search the collision-free path avoiding the set of obstacles $\mathcal{O}$ starting at the point P_1, ending at the point P_n and passing through the points $P_2, \ldots, P_{n-1}$. The achieved results allow construction of the collision-free path as piecewise quadratic spline. As we mention in the introduction, some applications may request C^1 continuity of the path because in mobile robotics a non smooth motions can cause slippage of the wheels.

Let the starting point P_1 and ending point P_2 be given. We find the set V containing such points C, that the Bézier curve $b_{P_1CP_2}$ is a collision-free path with respect to the set of obstacles $\mathcal{O}$. Using the results in previous section, we find the set $V_j(P_1, P_2)$ containing the admissible points C for collision-free path with respect to one obstacle O_j.

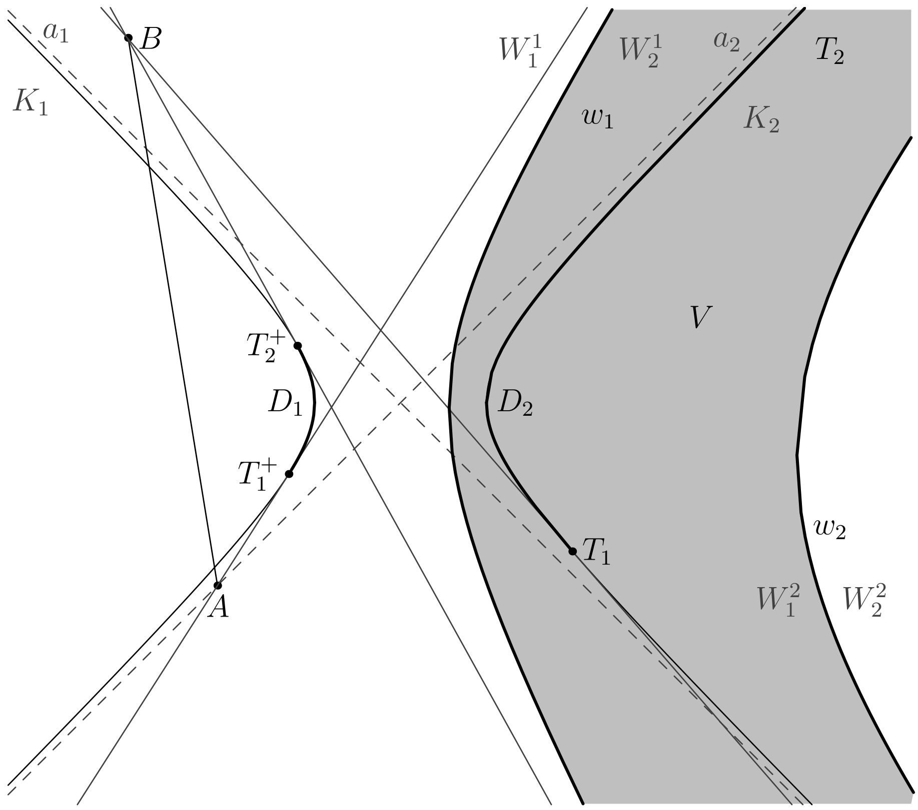

Figure 3: For the component K_1, the set of exterior points of contact $D^1_{ext} = \emptyset$ and the set of interior points of contact $D^1_{in} = \overset{\frown}{T_1^+ T_2^+}$. The set of points of contact $D^1 = D^1_{in}$ generates the curve $w_1 \subset \partial V_\rho(A, B)$. The curve w_1 divides the plane ρ into two regions W_1^1, W_2^1. Let the points $A, B \in W_1^1$. The set of admissible solutions for the component K_1 is $V^1(A, B) = W_2^1$. For the component K_2, since $T_2 = a_2^\infty$, the set of exterior points of contact $D^2_{ext} = \overset{\frown}{T_1 a_2^\infty}$ and the set of interior points of contact $D^2_{in} = \emptyset$. The set of points of contact $D^2 = D^2_{ext}$ generates the curve $w_2 \subset \partial V_\rho(A, B)$ and if the points $A, B \in W_1^2$, then $V^2(A, B) = W_1^2$. Finally, the set of admissible solutions for the conic section K is $V_\rho(A, B) = V^1(A, B) \cap V^2(A, B)$.

If we need to *avoid all obstacles simultaneously*, the point $C \in V = \bigcap_{j=1}^{m} V_j(P_1, P_2)$. The intersection can be computed using some of the standard algorithms for computing intersection of algebraic areas in CAD, e. g. cylindrical algebraic decomposition or vertical decomposition [2, 6]. Now, we may choose an arbitrary point $C \in V$ for obtaining collision-free path $b = b_{P_1CP_2}(t)$.

Let the starting point P_1 and ending point P_n be given. We need to find the system of sets $\mathcal{V} = \{V_1, \ldots, V_{n-1}\}$ such that each Bézier curve $b_{P_iC_iP_{i+1}}$ with $C_i \in V_i$ represent collision-free path between the points P_i and P_{i+1} with respect to all obstacles from $\mathcal{O}$. The path is constructed sequentially for each segment P_iP_{i+1}. We choose the set of points $\{C_1, \ldots, C_{n-1}\}$ such that C_i is an arbitrary point from $W_i = \bigcap_{j=1}^{m} V_j(P_i, P_{i+1})$. Then, the C^0-*continuous collision-free path* $b = \bigcup_{i=1}^{n-1} b_{P_iC_iP_{i+1}}(t)$. Computing of b for the set of three points $\{P_1, P_2, P_3\}$ and two obstacles $\{O_1, O_2\}$ is illustrated in the fig. 4 (a, b).

Let the set $\mathcal{V} = \{V_1, \ldots, V_{n-1}\}$, where each set V_i contains all C_i representing collision-free path between the points P_i and P_{i+1}. For obtaining the C^1-*continuous spline*, we pick the points $C_1, \ldots, C_{n-1}$. The decision must be compatible with the conditions of C^1 continuity [4]. Hence, the points C_{i-1} and C_i are centrally symmetric with respect to the point P_i for $i = 2, \ldots, n-1$. We denote by $\mathrm{Ref}_P(X)$ the reflection of a point X with respect to a point P . Note that by choosing C_1, we determine all remaining middle control points, because $C_2 = \mathrm{Ref}_{P_2}(C_1), \ldots, C_{n-1} = \mathrm{Ref}_{P_{n-1}}\mathrm{Ref}_{P_{n-2}} \ldots \mathrm{Ref}_{P_2}(C_1)$. So, we find the subset $W \subset V_1$ of such points C_1, that all reflected $C_i \in V_i$ for $i = 2, \ldots, n-1$. This ensures that the final spline is collision-free. Let the set $W = V_1 \cap \mathrm{Ref}_{P_2}(\mathrm{Ref}_{P_3}(\ldots(\mathrm{Ref}_{P_{n-1}}(V_{n-1}) \cap V_{n-2}) \cap \cdots \cap V_3) \cap V_2)$. For each point $C \in W$ the path $b = \bigcup_{i=1}^{n-1} b_{P_iC_iP_{i+1}}(t)$, where $C_1 = C$, $C_2 = \mathrm{Ref}_{P_2}(C), \ldots, C_{n-1} = \mathrm{Ref}_{P_{n-1}}\mathrm{Ref}_{P_{n-2}} \ldots \mathrm{Ref}_{P_2}(C)$, is collision-free with respect to obstacles $\mathcal{O}$ and C^1-continuous (see fig. 4 (c, d)).

We apply this study to the following example. Let the virtual agent starts at the point P_1 and needs to get to the point P_3 while passing through the point P_2. Moreover, it is supposed to avoid two obstacles O_1, O_2 represented by an ellipse and a hyperbola respectively. The path finding is illustrated in the fig. 4. Conic sections as bounding objects well represent a wide range of obstacles. For example, ellipses are suitable for buildings, trees and things and hyperbolas and parabolas for coast or areas with special features.

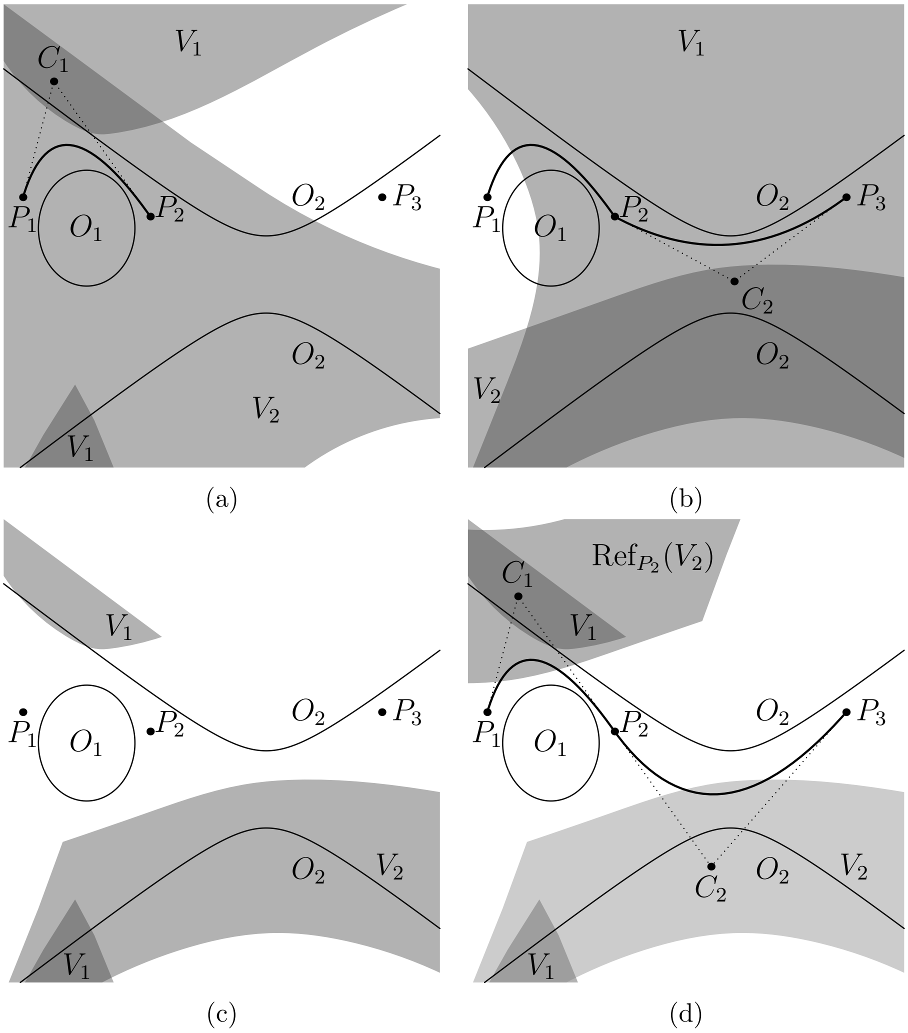

Figure 4: (a) First, we search the collision-free path between the points P_1, P_2. After finding the sets of admissible solutions V_1, V_2 for each obstacle separately, we pick a point $C_1 \in V_1 \cap V_2$. (b) Then, we find the sets V_1, V_2 for the points P_2, P_3 and choose the point $C_2 \in V_1 \cap V_2$. The final spline is C^0-continuous. (c) The set V_1 contains all such points C_1, that the curve $b_{P_1C_1P_2}$ is a collision-free segment. Similarly, the set V_2 contains all such points C_2 for the segment P_2P_3. (d) We have to pick the points C_1, C_2. Taking the point $C_1 \in V_1 \cap \mathrm{Ref}_{P_2}(V_2)$ and $C_2 = \mathrm{Ref}_{P_2}(C_1)$, the finial spline is C^1-continuous.

6 Cubic collision-free Bézier path

We look for the collision-free condition of the Bézier curves of higher degree than two. They provide more natural-looking path and they are more flexible while avoiding obstacles including non-planar curves. We start to study a conditions for planar cubic curves, using the knowledge and methods from the quadratic case.

Let $A, C, F, B \in \rho$ be the control points of the Bézier curve b_{ACFB}. Let A, B lie outside of the conic section K. Let the point F be arbitrary, but fixed. We need to find the *set of admissible solutions* $V_\rho(A, F, B)$ of such points $C \in \rho$, that the curve b_{ACFB} is collision-free with respect to K.

We say that the set $D \subset K$ is the *set of points of contact* between K and the set of all b_{ACFB} if for any point $X \in D$, there is a point C such that $C \in V_\rho^v(A, F, B)$ and $X \in b_{ACB} \cap K$. The exact shape of the set D is shown later.

We need to obtain the points C corresponding to the set D forming the boundary of the set $V_\rho(A, F, B)$. Let $\mathcal{P}(\rho)$ be the power set of the plane ρ. The map $\sigma : D \to \mathcal{P}(\rho)$ is called *boundary map* if for every $X \in D$ holds $\sigma(X) = \{C \in \rho \mid C \in V_\rho^v(A, F, B)$ and $X \in b_{ACFB} \cap K$ is the point of contact$\}$. Let the real numbers

$$\begin{aligned}\alpha &= (A - 3F + 2B)\boldsymbol{Q_K}X^\top,\\ \beta &= (F - A)\boldsymbol{Q_K}X^\top,\\ \gamma &= A\boldsymbol{Q_K}X^\top,\\ \delta &= -\gamma\end{aligned}$$

be the coefficients of the cubic function

$$R(t) = \alpha t^3 + 3\beta t^2 + 3\gamma t + \delta \tag{1}$$

for $A = [a_x, a_y, 1]$, $F = [f_x, f_y, 1]$, $B = [b_x, b_y, 1]$, $X = [x_0, y_0, 1]$. Then, the corresponding boundary map $\sigma : D \to \mathcal{P}(\rho)$ has the form

$$\sigma(X) = \left\{ \frac{b(t_0) - B_0^3(t_0)A - B_2^3(t_0)F - B_3^3(t_0)B}{B_1^3(t_0)},\ t_0 \in (0,1) \wedge R(t_0) = 0 \right\}.$$

For finding the set $V_\rho(A, F, B)$ is necessary to determine the number of roots of the equation (1) within the interval $\langle 0, 1\rangle$. We use the combination of Boudan-Fourier theorem [6], theory about Sturm-Habicht sequences [11] and the properties of the discriminant of cubic polynomial function.

6.1 Singular conic sections

We find the set of admissible points of the contact D for singular conic sections. If the conic section $K = p$, the set of admissible points of the contact $D = K$. Moreover, the boundary of the set of admissible solutions ∂V is a parallel line to the line p. Let $K = p \cup r$. The set of points of contact $D = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$ (in special case $S_p = S_r = V_Q$). From the previous lemma, the set ∂V consists of two half-lines parallel with p, resp. r, connected in the point C_u. If the conic section $K = \{V_Q\}$, then the set $D = \{V_Q\}$ and the boundary of the set of admissible solutions ∂V is one continuous curve.

6.2 Regular conic sections

For the set $D \subset K$ for regular conic sections, we determine the necessary conditions. We denote $P_A = A\boldsymbol{Q_K}X^\top, P_B = B\boldsymbol{Q_K}X^\top, P_F = F\boldsymbol{Q_K}X^\top$. The set of admissible points of the contact D is the subset of the union of the arcs $K_i \subset K$ for $i = 1, \dots, 4$, where

$$\begin{aligned}
K_1 &= \{X \in K\colon P_A \geq 0 \wedge P_B \geq 0\},\\
K_2 &= \{X \in K\colon P_A \leq 0 \wedge P_B \leq 0\},\\
K_3 &= \{X \in K\colon P_A \geq 0 \wedge P_B \leq 0 \wedge P_F^3 - P_A P_B^2 \geq 0\},\\
K_4 &= \{X \in K\colon P_A \leq 0 \wedge P_B \geq 0 \wedge P_F^3 - P_A P_B^2 \leq 0\}.
\end{aligned}$$

We also present some geometrical conditions, which are included in the sufficient conditions for the set D. We formulate some hypotheses about the shape of the set $V_\rho(A, F, B)$.

7 Main assets of the work

Our results solved the path planning problem for point-size robot in two or three dimensional space, because its configuration space can be identified with our working space for obstacles represented by regular quadrics. It can be also used for translational robots and all robots with two or three dimensional configuration space, when we describe a space C_{obs} by conic sections. The main benefit of the work is a mathematical model of path finding with proofs about existence and uniqueness, which involves a complete solution (we find all possible collision-free quadratic paths).

The finding of smooth collision-free path using Bézier curves was mentioned in the previous section. But the algorithms working with polynomials or Bézier curves are used as post processing, because they assume some linear collision-free path and they only smooth the path. Moreover, the algorithms provide only a numerical solution. The proposed method offers a direct analytical computation of all possible collision-free smooth paths without using sample-based planning algorithms. We assume an obstacle represented by a regular quadric and given start and end position of robot. We find all quadratic Bézier curves constituting the set of collision-free paths. One can use such a set for optimization of the sought path.

Sometimes the scene with the given obstacles is too complicated and the smooth collision-free path cannot be found directly. Then, the use of sample-based planning algorithms is necessary. But the obtained linear path is jerky, because it contains many redundant nodes which was generated randomly. In order to remove these nodes the path pruning techniques as in [10] are used. Our results can also form a path pruning algorithm, where the node v_i can be removed if there is a quadratic Bézier collision-free path between nodes v_{i-1} and v_{i+1}. Such an algorithm is more flexible comparing with a piecewise linear approach.

There is another use for the affine three dimensional Minkowski space typically determined by a light cone. Applications of Minkowski $\mathbb{R}^3_1$ space in CAGD are given in [9]. The set of all pointwise space-like curves is determined by the set V. The light cone represents a quadric. If we take two space-like points as a first and a last control point, we can find all quadratic pointwise space-like Bézier curves.

8 Conclusion

We conclude that the proposed goals were reached and the following results were achieved.

At first, we defined a working space. The three dimensional Euclidean space with regular quadric κ as obstacle and two given points A, B. The first goal was to find all quadratic Bézier paths starting at A and ending at B representing collision-free path with respect to an obstacle κ. This set of Bézier curves we represent by the set of corresponding middle control points $V(A, B)$. Because a quadratic curve lies in a plane (denote by ρ), we studied the set $V_\rho(A, B) = V(A, B) \cap \rho$ for each type of conic section $K = \kappa \cap \rho$. We determine the set of admissible points of contact $D \subset K$ between Bézier curves b_{ACB} and K, where $C \in \rho$. The set D consist of some arcs on K, where the polars of the points A, B play the key role. For the points A, B and contact point $X \in D$ is the Bézier curve uniquely defined. Using the map sigma $\sigma\colon D \to \rho$ the corresponding middle control point $C = \sigma(X)$ was obtained. We present the equations for σ and we proved also that thus obtained points C create the boundary $\partial V_\rho(A, B)$. The boundary consists of one ore two continuous curves, which divide the plane ρ in some areas. The set of admissible middle control points $V_\rho(A, B)$ consist of one or two areas depending on type of conic section.

The second goal was to find the quadratic spline representing collision-free path with respect to set of obstacles $O = \{O_1, \dots, O_m\}$ and passing through the points $P = \{P_1, \dots, P_n\}$. We solved the avoiding more obstacles simultaneously for $P = \{P_1, P_2\}$. At first, we determine the set $V_j(P_1, P_2)$ containing the admissible points C for collision-free path due to one obstacle O_j separately for $j = 1, \dots, m$. The set of admissible middle control points $V_\rho(P_1, P_2)$ is obtained as intersection of all the sets $V_j(P_1, P_2)$. If the set P contains more then two points, we create the spline as connection of quadratic Bézier paths between each pair P_iP_{i+1} for $i = 1, \dots, n-1$. For the creation of C^0-continuous spline it is necessary to find the sets $V_\rho(P_iP_{i+1})$ for each i. The existence of such spline is dependent on the existence of all the sets $V_\rho(P_iP_{i+1})$. For ensuring the C^1-continuity of the spline, we used the properties of the first derivative at the points of connection P_i for $i = 2, \dots, n-1$. Validity of the equalities $C_i = 2P_i - C_{i-1}$ for $i = 2, \dots, n-1$ is necessary for C^1-continuity of the spline. We used the fact that the points C_i, C_{i-1} are centrally symmetric with respect to the point P_i. So, if we choose the first $C_1 \in V\rho(P_1, P_2)$ then all remaining $C_2, \dots, C_{n-1}$ are uniquely determined. We showed that if the point C_1 is chosen from the set $W = V_1 \cap \mathrm{Ref}_{P_2}(\mathrm{Ref}_{P_3}(\dots(\mathrm{Ref}_{P_{n-1}}(V_{n-1}) \cap V_{n-2}) \cap \dots \cap V_3) \cap V_2)$ where $V_i = V_\rho(P_i, P_{i+1})$, then the spline is collision-free with respect to set of obstacles O. The existence of such spline is dependent on the $W \neq \emptyset$.

The third goal was to apply the observations from quadratic paths on the collision-free paths represented by planar cubic Bézier curves and to formulate some theorems and hypothesis about them. We focus on planar cubic Bézier paths with fixed control point F. We solve the situation in the plane ρ with respect to $K = \kappa \cap \rho$. We determine the map $\sigma\colon D \to \rho$ for cubic Bézier curves. From the equations of this map, we derive the necessary conditions for the set D as arcs $K_1, \dots, K_4$ on K. Again, the key role is played by polars from the points A, B and the points where discriminant of equation (1) vanishes. For singular K, the set $V_\rho(A, F, B)$ of admissible points C consist of one or two areas similarly as in quadratic case. For regular K, we formulate sufficient geometrical conditions for the set D in some special cases. For other cases, we formulate some hypothesis. While searching for the conditions, the property of Bézier curves that they lie in the convex hull of their control points was very helpful. We formulate some hypotheses about the set $V_\rho(A, F, B)$, because they work very well in a lot of examples.

References

[1] J. Barraquand and J.-C. Latombe. Robot Motion Planning: A Distributed Representation Approach. *International Journal of Robotics Research*, 10(6):628–649, December 1991.

[2] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[3] M. Berger. *Geometry I, II*. Universitext. Springer-Verlag, Berlin, 1987. Translated from the French by M. Cole and S. Levy.

[4] M. Berger and B. Gostiaux. *Differential geometry: manifolds, curves, and surfaces*, volume 115 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988. Translated from the French by S. Levy.

[5] R. Bix. *Conics and cubics*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2006. A concrete introduction to algebraic curves.

[6] J.-D. Boissonnat and M. Teillaud, editors. *Effective computational geometry for curves and surfaces*. Mathematics and Visualization. Springer-Verlag, Berlin, 2007.

[7] M. Elbanhawi and M. Simic. Sampling-based robot motion planning: A Review. *IEEE Access*, 2:56–77, 2014.

[8] G. Farin, J. Hoschek, and M.-S. Kim, editors. *Handbook of computer aided geometric design*. North-Holland, Amsterdam, 2002.

[9] R. T. Farouki. *Pythagorean-hodograph curves: algebra and geometry inseparable*, volume 1 of *Geometry and Computing*. Springer, Berlin, 2008.

[10] R. Geraerts and M. H. Overmars. Creating High-quality Paths for Motion Planning. *The International Journal of Robotics Research*, 26(8):845–863, 2007.

[11] L. González-Vega and G. Trujillo. Multivariate Sturm-Habicht sequences: real root counting on n-rectangles and triangles. *Rev. Mat. Univ. Complut. Madrid*, 10(Special Issue, suppl.):119–130, 1997. Real algebraic and analytic geometry (Segovia, 1995).

[12] J. Hoschek and D. Lasser. *Fundamentals of computer aided geometric design*. A K Peters Ltd., Wellesley, MA, 1993. Translated from the 1992 German edition by L. L. Schumaker.

[13] K. G. Jolly, R. Sreerama Kumar, and R. Vijayakumar. A Bézier curve based path planning in a multi-agent robot soccer system without violating the acceleration limits. *Robotics and Autonomous Systems*, 57(1):23–33, 2009.

[14] E. Kunz. *Introduction to plane algebraic curves.* Birkhäuser Boston Inc., Boston, MA, 2005. Translated from the 1991 German edition by R. G. Belshoff.

[15] J.-C. Latombe. *Robot Motion Planning.* Kluwer, Boston, MA, 1991.

[16] S. M. LaValle. *Planning Algorithms.* Cambridge University Press, Cambridge, U.K., 2006. Available at http://planning.cs.uiuc.edu/.

[17] D. Marsh. *Applied geometry for computer graphics and CAD.* Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, second edition, 2005.

[18] J. Pan, L. Zhang, and D. Manocha. Collision-Free and Curvature-Continuous Path Smoothing in Cluttered Environments. In *Robotics: Science and Systems VII, University of Southern California, Los Angeles, CA, USA, June 27-30, 2011*, 2011.

[19] J. Pan, L. Zhang, and D. Manocha. Collision-free and smooth trajectory computation in cluttered environments. *I. J. Robotic Res.*, 31(10):1155–1175, 2012.

[20] H. Prautzsch, W. Boehm, and M. Paluszny. *Bézier and B-spline techniques.* Mathematics and Visualization. Springer-Verlag, Berlin, 2002.

[21] K. Yang, D. Jung, and S. Sukkarieh. Continuous curvature path-smoothing algorithm using cubic Bézier spiral curves for non-holonomic robots. *Advanced Robotics*, 27(4):247–258, 2013.

[22] K. Yang and S. Sukkarieh. 3D Smooth Path Planning for a UAV in Cluttered Natural Environments. *Proc. IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 794–800, 2008.

[23] K. Yang and S. Sukkarieh. An Analytical Continuous-Curvature Path-Smoothing Algorithm. *IEEE Transactions on Robotics*, 26(3):561–568, 2010.

Publications

Conference and journal publications

Pokorná, B., Chalmovianský, P.: *Collision-free piecewise quadratic spline with regular quadratic obstacles.* In: KoG. *(in press)*

Bátorová, M., Chalmovianský, P., Pokorná, B., Valíková, M.: *Singular Point of Curves – Structure, Visualization and Application in Geometric Modelling.* In: Information Technology Applications / Scientific Journal, vol. 2–2013, 2013. pp. 5–22.

Pokorná, B., Chalmovianský, P.: *Planar cubic space-like Bézier curves in three dimensional Minkowski space.* In: Proceedings of Symposium on Computer Geometry, vol. 21. ISBN 978-80-227-3798-2. Bratislava : Slovenská technická univerzita, 2012. pp. 93–98.

Pokorná, B., Chalmovianský, P.: *Quadratic space-like Bézier curves in the three dimensional Minkowski space.* In: Proceedings of Symposium on Computer Geometry, vol. 20. ISBN 978-80-227-3580-3. Bratislava : Slovenská technická univerzita, 2011. pp. 104–110.

Pokorná, B.: *Bézier curves and their properties in Minkowski space (in Slovak; Bézierove krivky a ich vlastnosti v Minkowského priestore).* In: Proceedings of the 50th Anniversary of Foundation of the Department of Geometry on Faculty of Natural Sciences of Comenius University: Seminar Proceedings. ISBN 978-80-8147-000-4. Bratislava : Knižničné a edičné centrum FMFI UK, 2012. pp. 47.

Gallusová, B.: *Bézier curves and their properties in Minkowski space (in Slovak; Bézierove krivky a ich vlastnosti v Minkowského priestore).* In: Proceedings of Student Scientific Conference 2010. Ostrava: Technical University of Ostrava, 2010. pp. 22–23.

Gallusová, B.: *Bézier curves and their properties in Minkowski space (in Slovak; Bézierove krivky a ich vlastnosti v Minkowského priestore).* In: Proceedings of Students Scientific Conference FMFI UK. ISBN 978-80-89186-68-6. Bratislava : Fakulta matematiky, fyziky a informatiky UK, 2010. pp. 30–40.

Research projects

Space-like cubic Bézier curves in Minkowski space. UK/328/2012. 2012. Comenius University grant (€1000).
Principal investigator: RNDr. Barbora Pokorná.

Space-like Bézier curves in three dimensional Minkowski space. UK/279/2011. 2011. Comenius University grant (€830).
Principal investigator: RNDr. Barbora Pokorná.

Tools of Geometric Modeling. VEGA 1/0730/09. 2011.
Principal investigator: doc. RNDr. Miloš Božek, PhD.

Theses

Pokorná, B.: *Quadratic space-like Bézier curves in three dimensional Minkowski space*, Project of Dissertation Thesis, 2012.

Pokorná, B.: *Quadratic Bézier curves in three dimensional Minkowski space (in Slovak; Kvadratické Bézierove krivky v trojrozmernom Minkowského priestore)*, Rigorous Thesis, 2011.

Gallusová, B.: *Bézier curves and their properties in Minkowski space (in Slovak; Bézierove krivky a ich vlastnosti v Minkowského priestore)*, Master Thesis, 2010.

Awards

Rector's Award, 2010.

MSc. degree with Honors, 2010.

ŠVOČ (Student Scientific Conference, Czech and Slovak international round), Chair's Award, 2010.

ŠVK (Student Scientific Conference), Winner, 2010.