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Mgr. Csilla Krommerová

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Dynamic portfolio optimization with risk management and strategy constraints

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Predkladateľ: Mgr. Csilla Krommerová
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave
Mlynská dolina, 842 48 Bratislava

Školiteľ: Doc. Mgr. Igor Melicherčík, PhD.
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave

Oponenti:

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na Fakulte matematiky, fyziky a informatiky Univerzity Komenského v Bratislave, Mlynská dolina, 842 48 Bratislava

Predseda odborovej komisie
Prof. RNDr. Marek Fila, DrSc.
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave
Mlynská dolina, 842 48 Bratislava

1 Introduction

The problem of maximizing the expected utility over a given time horizon is one of the most frequently examined problems in financial mathematics. One can achieve the maximum expected utility by choosing the proper portfolio strategy, i.e. by optimal allocation of the available funds among risky and risk-free assets.

The problem was first examined by P. A. Samuelson. In his work [13], Samuelson considers the return on the risky asset to be stochastic with a general probability distribution. He presents the model in a discrete form and interprets it as a problem of dynamic stochastic programming, solving the Bellman equation. He states that for power utility functions, the optimal portfolio strategy is constant over time. Merton [7] confirms the results of Samuelson for a continuous-time case, assuming that the returns on the risky assets are generated by a Brownian motion.

In his work, Nutz [8] expands the power utility maximization problem using a special case, when the prices follow the exponential Lévy process. His approach to the problem is also based on the construction of the corresponding Bellman equation. Nutz proves that the results of Samuelson and Merton hold in this case too. Additionally, Nutz examines the case when the portfolio strategy is constrained by a fixed convex set and shows that in such case the optimal portfolio strategy is also constant. In [9], Nutz considers stochastic portfolio constraints and shows that the portfolio strategy can be obtained as the argmax of a predictable function.

No portfolio with risky assets guarantees any return. The aim of the portfolio insurance is to limit the losses and simultaneously to allow the participation on the rising market. The idea of insuring the portfolio against losses was first introduced by H. Leland and M. Rubinstein in 1976. They developed the option based portfolio insurance, also referred to as OBPI. The OBPI consists of a risky asset and a put option written on it. The strike price of the put option represents the floor such that the value of the investment at the maturity is higher than the floor with 100% probability.

There is a possibility that the required put option is not available on the market. By Leland and Rubinstein [4], in such case one can synthesize the put option with a replication portfolio that consists of the underlying asset and a risk-free bond. Using the replication portfolio the OBPI becomes dynamic, so that one can guarantee the discounted level of the floor at any time from the beginning until the maturity.

In 1986, Perold [10] introduced another type of dynamic portfolio insurance, the constant proportion portfolio insurance, also referred to as CPPI (see also [11]). The CPPI agent first determines the floor under which the portfolio is not allowed to fall at the terminal date. At each time he calculates the difference between the discounted level of the floor and the actual value of the portfolio, the so-called cushion. The exposure to the risky assets is calculated as the cushion multiplied by a predefined constant multiplier. Both the floor and the multiplier are the characteristics of the agents risk-tolerance.

While the OBPI and CPPI methods require a guaranteed floor with probability one, the Value-at-Risk based risk management guarantees the floor with a given probability less than one. Basak and Shapiro [1] introduced the power utility optimization model using the Value-at-Risk based risk management (also called VaR-RM).

2 Goals of the thesis

Even though both the optimal portfolio selection and the portfolio insurance were examined by many scientists, it still offers many research opportunities. The aim of this work is to bring together these two areas, specifically, we investigate how to insure the portfolio when convex constraints are imposed on the portfolio strategy. We intend to provide either optimal or admissible solutions for the problem of dynamic portfolio optimization with risk management and strategy constraints.

We specify the convex constraints representing the case when short-selling of both the risky and risky-free assets is prohibited. Our goal is to investigate two main areas

- the portfolio insurance with a guaranteed floor in the constrained model,
- the portfolio insurance with a partially guaranteed floor in the constrained model,

provide different methods of solution and compare them based on their certainty equivalents.

3 Methods and results

3.1 Economic settings

Let $T > 0$ represent the time horizon and let the triplet $(\Omega, \mathcal{F}, P)$ represent the probability space. We use d risky assets and one risk-free bond to construct our portfolio.

For a given quantity, we use the upper index $i = 1, 2, ..., d$ to represent a particular asset and the lower index $t \in \langle 0, T \rangle$ to express the time dependence.

We denote the expected return on the asset i by μ^i, the positive definite volatility matrix by $\sigma = \{\sigma^{ij}, i = 1, ..., d, j = 1, ..., d\}$, the covariance matrix by $c^R = \sigma\sigma^\top$ and the risk-free interest rate by r. We consider these parameters to be constant over the time.

Let $w_t = (w_t^1, w_t^2, ..., w_t^d)^\top$ be an $\mathbb{R}^d$-valued Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then the prices of the risky assets and the non-risky bond follows

$$dS_t^i = S_t^i[\mu_i dt + \sigma_i dw_t^i], \qquad \text{for } i = 1, 2, ..., d, \tag{1}$$

$$dB_t = B_t r dt. \tag{2}$$

We define the portfolio strategy as $\beta_t = (\beta_t^1, \beta_t^2, ..., \beta_t^d)^\top$, where β_t^i represents the proportion of the total wealth invested in the i-th asset at time t. For simplicity we fix the initial capital X_0. The wealth process then follows

$$dX_t = X_t[r + \beta_t^\top(\mu - r\mathbf{1})]dt + X_t\beta_t^\top \sigma dw_t, \tag{3}$$

where $\mathbf{1} = (1, 1, ..., 1)^\top$.

The existence of the state price density process ξ_t ensures the market completeness (under no-arbitrage). The stochastic differential equation for ξ_t is given as

$$d\xi_t = -\xi_t[rdt + \kappa^\top dw_t], \tag{4}$$

where $\kappa = \sigma^{-1}(\mu - r\mathbf{1})$ is the market price of the risk process and is also considered to be constant over time. In all cases we consider the portfolio to be self-financing

$$E[\xi_T X_T] \leq \xi_0 X_0,$$

i.e. after the initial investment, no further investments are needed (the assumption of zero net investments), and buying or selling one type of asset is balanced by selling or buying other assets (the principle of self-financing).

The agent strives to utilize the expected terminal wealth $U(X_T)$. The utility function U is assumed to be increasing, concave and twice continuously differentiable. In our work, we focus on the power utility functions of the form

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0. \tag{5}$$

We exclude the case when $\gamma = 1$, as in this case the utility function is logarithmic.

By Prigent [12], the power utility functions have a constant Arrow-Pratt measure of relative risk-aversion in the form

$$R(W_T) = -W_T \frac{U(W_T)''}{U(W_T)'} = \gamma.$$

Mehra and Prescott [6] state that a reasonable relative risk-aversion takes values between $\gamma \in \langle 2, 10 \rangle$. The higher the parameter of the risk aversion is, the more conservative the agent is.

Note that in the literature, the power utility function can also be referred to as isoelastic function or CRRA (Constant Relative Risk Aversion) function.

3.2 Power utility maximization

When using the power utility function and assuming no strategy constraints, our aim is to find the optimal portfolio strategy that maximizes the expected utility from the terminal wealth, i.e.

$$\max_{\beta} E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right], \tag{6}$$

where we maximize through all dynamic strategies β. By Nutz [8], the optimal portfolio strategy is the argmax of a deterministic function

$$\eta(\beta) = r + \beta^T(b^R - r\mathbf{1}) - \frac{\gamma}{2}\beta^T c^R \beta \tag{7}$$

and can be expressed as

$$\hat{\beta} = \frac{1}{\gamma}(c^R)^{-1}(\mu - r\mathbf{1}). \tag{8}$$

Let $\mathcal{S} \subseteq \mathbb{R}^d$ be the set of constraints imposed on the agent. Then the set of admissible strategies according to the initial wealth X_0 is

$$\mathcal{A}(X_0) := \{\beta : X_t > 0 \quad \text{and} \quad \beta_t \in \mathcal{S} \quad \text{for all} \quad t \in \langle 0, T \rangle\}.$$

In case of fixed X_0, we simply write $\mathcal{A}$ instead of $\mathcal{A}(X_0)$ and we optimize

$$\max_{\beta \in \mathcal{A}} E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]. \tag{9}$$

Theorem 1 ([8], Theorem 3.2.)**.** *Assume that $\mathcal{S}$ is convex and there is no arbitrage on the market. Then, there exists an optimal strategy $\hat{\beta}$ such that $\hat{\beta}$ is a constant vector and is characterized by*

$$\hat{\beta} \in \arg\max_{\beta \in \mathcal{S}} \eta(\beta), \tag{10}$$

where $\eta(.)$ is given in (7).

4 Portfolio insurance with guaranteed floor

The main idea of insuring the portfolio against losses is to guarantee a minimum return and simultaneously allow the portfolio to participate on the rising market.

The OBPI strategy consist of a portfolio covered by a put option written on it. The put option has the same maturity T as the portfolio and its strike price $\underline{W}$ is the predefined floor. The basic overview of OBPI can be found in [2].

Let the risky portfolio X, invested in d risky assets and a non-risky bond follow the process

$$dX_t = X_t \mu_X dt + X_t \sigma_X dw_t, \tag{11}$$

where $\mu_X = r + \beta^\top(\mu - r)$ is the drift of the portfolio, $\sigma_X = \sqrt{\beta^\top c^R \beta}$ is the volatility of the portfolio and w_t is a one-dimensional Brownian motion.

Let V_t^{put} be the price of the put option and V_t^{call} be the price of the call option with maturity T and strike price $\underline{W}$ at time $t \in \langle 0, T\rangle$. The value of the insured portfolio W_t at time t is given as

$$\begin{aligned} W_t &= X_t + V_t^{put} \\ &= \underline{W} e^{-r(T-t)} + V_t^{call} \end{aligned}$$

due to the put-call parity. One can see that the value of the insured portfolio W_t is always above the deterministic level $\underline{W} e^{-r(T-t)}$ at any time t.

Using the Black-Scholes pricing, the prices of V_t^{put} and V_t^{call} at time t can be calculated as

$$\begin{aligned} V_t^{put} &= \underline{W} e^{-r(T-t)} \Phi\left(-d_2(\underline{W})\right) - X_t \Phi\left(-d_1(\underline{W}\right) \\ V_t^{call} &= X_t \Phi\left(d_1(\underline{W})\right) - \underline{W} e^{-r(T-t)} \Phi\left(d_2(\underline{W})\right), \end{aligned} \tag{12}$$

with

$$\begin{aligned} d_1(\underline{W}) &= \frac{\ln \frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \\ d_2(\underline{W}) &= d_1 - \sigma_X \sqrt{T-t}, \end{aligned}$$

where $\Phi(.)$ is the standard normal distribution function.

Possible difficulties might occur when the desired put option cannot be found on the market. In such case the put option can be synthesized by a replication portfolio invested in the risk-free asset and the underlying portfolio. The replication portfolio should have the same characteristics as the put option (e.g. the value, payoff and risk).

The replication portfolio at time t can be expressed as

$$V_t = \varphi_t X_t + \psi_t B_t, \tag{13}$$

where $\varphi_t = \dfrac{\partial V_t}{\partial X_t}$ is the so called delta of the option, in other words the sensitivity of the option-value on the value of the underlying portfolio. The delta of the put option φ_t can be computed as

$$\varphi_t = \Phi(d_1(\underline{W})) - 1 \tag{14}$$

and one can easily see that $-1 < \varphi_t < 0$, for every t (see [**?**]).

Then the value of the insured portfolio can be expressed as

$$\begin{aligned} W_t &= X_t + V_t^{put} \\ &= X_t + \varphi_t X_t + \psi_t B_t \\ &= (1+\varphi_t) X_t + \psi_t B_t. \end{aligned}$$

Because the portfolio weights are calculated as

$$weight^i = \frac{money\ invested\ in\ the\ asset\ i}{total\ money\ invested},$$

the new portfolio strategy can be expressed as

$$\theta_t^i = \frac{(1+\varphi_t)\beta^i X_t}{W_t}, \qquad i = 1, ..., d. \tag{15}$$

Subsequently, the portfolio process follows

$$dW_t = W_t \mu_W dt + W_t \sigma_W dw_t, \tag{16}$$

where the drift is $\mu_W = r + \theta_t^\top(\mu - r\mathbf{1})$, the volatility is $\sigma_W = \sqrt{\theta_t^\top c^R \theta^\top}$ and w_t is a one-dimensional Brownian motion.

The OBPI ensures that the terminal wealth is always above the floor

$$\begin{aligned} W_T &= X_T + V_T^{Put} \\ &= X_T + \max(0, \underline{W} - X_T) \\ &= \max(X_T, \underline{W}). \end{aligned}$$

4.1 OBPI in the unconstrained model

The portfolio manager aims to maximize the utility from the expected terminal wealth of the insured portfolio

$$\begin{aligned} &\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad &W_T \geq \underline{W}, \end{aligned} \tag{17}$$

where the maximum is taken through all dynamic strategies θ. Note that in order to avoid immediate arbitrage situations, the floor must satisfy the condition $\underline{W} < W_0 e^{rT}$, where $W_0 > 0$ is the initial amount invested in the portfolio insured with OBPI.

Theorem 2. *The optimal portfolio strategy for the problem (17) is*

$$\hat{\theta}_t = \frac{1}{\gamma}[c^R]^{-1}(\mu - r\boldsymbol{1})\frac{(1+\varphi_t)X_t}{W_t},$$

where X_t is given by (11), and W_t follows (16). The fraction of wealth invested in stocks can be expressed as

$$\hat{\theta}_t = q_t\hat{\beta}, \tag{18}$$

where $\hat{\beta}$ is the optimal portfolio strategy of the uninsured model without constraints (6), calculated by (8) and

$$q_t = \frac{(1+\varphi_t)X_t}{W_t}. \tag{19}$$

4.2 OBPI in the constrained model

Now, let us consider a portfolio with convex constraints on the portfolio strategy and simultaneously it is insured by a put option. Mathematically, our model can be written as

$$\begin{aligned} &\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad &W_T \geq \underline{W}, \\ &\mathcal{C} = \{\theta^i \geq 0,\ i = 1, 2, ..., d; \sum \theta^i \leq 1\}. \end{aligned} \tag{20}$$

Theorem 3. *Let $\hat{\beta}$ be the optimal portfolio strategy for the portfolio with convex constraints, computed as*

$$\hat{\beta} = \arg\max_{\beta \in \mathcal{C}}\ \beta^{\top}(\mu - r\boldsymbol{1}) - \frac{1}{2}\gamma\beta^{\top}c^R\beta,$$

where $\mathcal{C}$ is given in problem (20). Let X_t follow (11) and W_t follow (16). Let φ_t be the delta of the put option calculated by (14). Then the portfolio strategy

$$\theta_t = (1+\varphi_t)\hat{\beta}\frac{X_t}{W_t} \tag{21}$$

is admissible for problem (20).

Corollary 1. *Let the solution $\hat{\beta}$ computed by (8) be optimal for the problem (6). In case that the optimal solution $\hat{\beta}$ with no constraints on the portfolio strategy satisfies $\hat{\beta}^i \geq 0$ for $i = 0, ..., d$ and $\sum_{i=1}^{d} \hat{\beta}^i \leq 1$, the portfolio strategy $\theta_t = \dfrac{(1+\varphi_t)X_t}{W_t}\hat{\beta}$ is optimal for the Problem (20).*

4.3 Alternative method in the constrained model

Now, we provide an alternative strategy for the problem (20). Denote the risky asset by Xa with a given constant portfolio strategy βa and let the insured portfolio be represented by Wa with a given dynamic portfolio strategy θa_t.

Let the set of constraints restrict only the short positions of the risky assets. Then the set of admissible strategies for βa can be described as $\mathcal{C}a = \{\beta a^i \geq 0,\ i = 1, 2, ..., d\}$. We determine the optimal portfolio strategy $\hat{\beta}a$ of the risky asset Xa from (10), using $\mathcal{S} = \mathcal{C}a$. The volatility of the risky asset Xa is $\sigma_{Xa} = \sqrt{\hat{\beta}a^{\top} c^R \hat{\beta}a}$. Then Xa follows the process

$$dXa_t = Xa_t[r + \hat{\beta}a^{\top}(\mu - r\mathbf{1})]dt + Xa_t\,\sigma_{Xa}dw_t, \tag{22}$$

where w_t is a one-dimensional Brownian motion.

The insured portfolio Wa consists of the risky asset Xa and a put option written on it. Its value at time t can be expressed as

$$Wa_t = Xa_t + Va_t^{put},$$

where Va_t^{put} is the value of the put option at time t. Let $\underline{W}$ be the strike price of the put option and T be the maturity. Because the particular put option might not be available on the market, we synthesize it.

At time t, the delta of the put option can be calculated as

$$\varphi a_t = \Phi\left(\frac{\ln\dfrac{Xa_t}{\underline{W}} + \left(r + \dfrac{\sigma_{Xa}}{2}\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) - 1$$

and the candidate for the portfolio strategy is

$$h_t = (1 + \varphi a_t)\, \hat{\beta a} \frac{Xa_t}{Wa_t}.$$

The problem (20) requires that the sum of the portfolio weights does not access the upper bound 1, therefore we define the new portfolio strategy as

$$\theta a_t = \begin{cases} (1 + \varphi a_t)\, \hat{\beta a} \dfrac{Xa_t}{Wa_t} & \text{if} \quad \sum_{i=1}^{d} h_t^i \le 1, \\ \\ \dfrac{(1 + \varphi a_t)\, \hat{\beta a} \dfrac{Xa_t}{Wa_t}}{\sum_{i=1}^{d} h_t^i} & \text{if} \quad \sum_{i=1}^{d} h_t^i \ge 1. \end{cases}$$

Portfolio Wa then follows

$$dWa_t = Wa_t[r + \theta a_t^\top (\mu - r\mathbf{1})]dt + Wa_t \sqrt{\theta a_t^\top c^R\, \theta a_t} dw_t. \tag{23}$$

Note that since $\hat{\beta a} \in Ca$, the portfolio strategy $\theta a_t \ge \mathbf{0}$.

Theorem 4. *Let* $Ca = \{\, \beta a^i \ge 0,\ i = 1, 2, ..., d\}$ *and* $\hat{\beta a}$ *is calculated by*

$$\hat{\beta a} = \arg \max_{\beta a \in Ca} \ \beta a^\top (\mu - r\boldsymbol{1}) - \frac{1}{2}\gamma\, \beta a^\top c^R\, \beta a. \tag{24}$$

Let Wa_t *be the value of the portfolio at time* t*. For* $Wa_t \ge \underline{W} e^{-r(T-t)}$*, we define the portfolio strategy* θa_t *as*

$$\theta a_t = \begin{cases} (1 + \varphi a_t)\, \hat{\beta a} \frac{Xa_t}{Wa_t} & \textit{if} \quad \sum_{i=1}^{d} h_t^i \le 1, \\ \\ \dfrac{(1 + \varphi a_t)\, \hat{\beta a} \frac{Xa_t}{Wa_t}}{\sum_{i=1}^{d} h_t^i} & \textit{if} \quad \sum_{i=1}^{d} h_t^i > 1, \end{cases} \tag{25}$$

where $h_t^i = (1 + \varphi a_t)\, \hat{\beta a}^i \frac{Xa_t}{Wa_t}$*. If* $Wa_0 \ge \underline{W} e^{-rT}$*, then* θa_t *is admissible for the problem (20) and* Wa_t *satisfies (23). Moreover,* $Wa_t \ge \underline{W} e^{-r(T-t)}$ *for all* $t \ge 0$ *with probability 1.*

4.4 Sensitivity analysis

Let us now examine the portfolio performance of the OPBI in the constrained model and the portfolio performance of the alternative method for different settings.

r	C	Ca
1%	1.03372	1.03393
2%	1.05016	1.05028
4%	1.07097	1.07091

γ	C	Ca
3	1.05437	1.05655
5	1.05016	1.05028
8	1.04391	1.04386

$\underline{W}$	C	Ca
0.98	1.06213	1.06240
1	1.05016	1.05028
1.01	1.04071	1.04078

Table 1: Certainty equivalents of the OBPI in the constrained model and of the alternative method.

We compare the two methods by changing the values of the risk-free interest rate r, the parameter of the power utility function γ and the floor $\underline{W}$. We use three risky assets and one risk-free bond to examine whether one method dominates the other one. We change only one parameter at the time, the remaining variables are kept fixed. We set the initial wealth $W_0 = 1$, the maturity $T = 1$, the vector of the expected returns $\mu = (0.06626, 0.1113, 0.1625)$ and the covariance matrix as $c^R = \begin{pmatrix} 0.02155 & 0.00825 & 0.00749 \\ 0.00825 & 0.01517 & 0.01190 \\ 0.00749 & 0.01190 & 0.05011 \end{pmatrix}$, based on data analysis.

By default, we set the risk-free interest rate $r = 2\%$, the power utility parameter $\gamma = 5$ and the floor $\underline{W} = 1$.

Table 4.4 compares the certainty equivalents of the OBPI in the constrained model and of the alternative method calculated as

$$C = \left((1-\gamma)E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \quad \text{and} \quad Ca = \left((1-\gamma)E\left[\frac{Wa_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}.$$

We see that there is no exact answer whether one should choose the OBPI in the constrained model or the alternative method. In other words, the OBPI is not optimal in the constrained model.

When changing the interest rate r or the parameter of the absolute risk-aversion γ, none of the methods dominate the other one. When changing the floor $\underline{W}$, in our specific settings, the alternative method dominates the OBPI in constrained model.

5 Portfolio insurance with a partially guaranteed floor

In this section, we allow the portfolio to fall under the guaranteed floor with a given probability. We show that the Value-at-Risk based risk management in the constrained model is not admissible and we provide an alternative admissible strategy to it, the portfolio insurance with spreads.

5.1 Value-at-Risk based risk management

In case of insuring the portfolio with a put option, the terminal value W_T of the portfolio does not fall under the predefined floor, i.e. $W_T \geq \underline{W}$ with the probability of 100%. Now, let us investigate the case of relaxing the condition

$$P(W_T \geq \underline{W}) = 1$$

and consider instead the probability of falling under the predefined floor to be greater than $1-\alpha$, i.e.

$$P(W_T \geq \underline{W}) \geq 1-\alpha. \tag{26}$$

Inequality (26) represents the so-called Value-at-Risk constraint.

Note that for $\alpha = 1$ the investor behaves as a benchmark agent, who does not consider any risk management. If $\alpha = 0$, the investor behaves as a portfolio insurer (securing with put options). In such case the terminal wealth will exceed the "floor" at all states.

5.2 VaR-RM in the unconstrained model

Our goal is to maximize the expected utility from the terminal wealth under VaR-RM

$$\begin{aligned} & \max_{\theta} E[U(W_T)] \qquad (27) \\ s.t. \qquad & P(W_T \geq \underline{W}) \geq 1-\alpha, \end{aligned}$$

where we maximize through all dynamic strategies θ and the initial is given as W_0.

The next proposition introduces the optimal wealth and portfolio strategy assuming that the utility function is isoelastic.

Proposition 1. *[[1], Proposition 3.] Assume that* $U(W) = \dfrac{W^{1-\gamma}}{1-\gamma}$ *for* $\gamma > 0$ *and that* r *and* κ *are constants. Then*

i) The optimal wealth at time t *is given by*

$$\begin{aligned} W_t^{VaR} \quad &= \frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} - \\ &- \left[\frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{VaR}(\underline{\xi})\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{VaR}(\underline{\xi})\right) \right] \\ &+ \left[\frac{e^{\Gamma_t^{VaR}}}{(y\xi_t)^{\frac{1}{\gamma}}} \Phi\left(-d_1^{VaR}(\overline{\xi})\right) - \underline{W} e^{-r(T-t)} \Phi\left(-d_2^{VaR}(\overline{\xi})\right) \right], \end{aligned}$$

where $\Phi(.)$ *is the standard-normal cumulative distribution function,* $y \geq 0$ *solves* $E[\xi_T W_T^{VaR}(y)] = \xi_0 X_0$ *and*

$$\underline{\xi} = \frac{1}{y\,\underline{W}^{\gamma}},$$

$$\Gamma_t^{VaR} = \frac{1-\gamma}{\gamma}\left(r + \frac{\|\kappa\|^2}{2}\right)(T-t) + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\|\kappa\|^2}{2}(T-t),$$

$$d_2^{VaR}(x) = \frac{\ln\dfrac{x}{\xi_t} + \left(r - \dfrac{\|\kappa\|^2}{2}\right)(T-t)}{\|\kappa\|\sqrt{T-t}},$$

$$d_1^{VaR}(x) = d_2^{VaR}(x) + \frac{1}{1-p}\|\kappa\|\sqrt{T-t}.$$

ii) The fraction of wealth invested in stocks is

$$\theta_t^{VaR} = q_t^{VaR}\hat{\beta},$$

where $\hat{\beta}$ *is the portfolio strategy of the benchmark agent, calculated by (8) and*

$$\begin{aligned} q_t^{VaR} &= 1 - \frac{\underline{W}e^{-r(T-t)}\left[\Phi\left(-d_2^{VaR}(\underline{\xi})\right) - \Phi\left(-d_2^{VaR}(\overline{\xi})\right)\right]}{W_t^{VaR}} \\ &\quad + \frac{\gamma\left(\underline{W} - \underline{\underline{W}}\right)e^{-r(T-t)}\phi\left(d_2^{VaR}(\overline{\xi})\right)}{W_t^{VaR}\|\kappa\|\sqrt{T-t}}, \end{aligned}$$

where $\phi(.)$ *is the standard-normal probability function.*

iii) The exposure to risky assets relative to the benchmark is bounded below, namely $q_t^{VaR} \geq 0$ *and*

$$\lim_{\xi_t \to 0} q_t^{VaR} = \lim_{\xi_t \to \infty} q_t^{VaR} = 1.$$

Note that W_t^{VaR} is a decreasing function of ξ_t for all $t \in \langle 0, T \rangle$.

The advantage of focusing on power utility functions is that knowing the optimal strategy $\hat{\beta}$ of the benchmark agent and the ratio q^{VaR}, which can be calculated from the model settings, one can easily determine the optimal strategy θ_t^{VaR} of the maximizing problem under VaR-RM at each time t.

5.3 VaR-RM in the constrained model

Basak and Shapiro [1] derived the VaR-RM model for the portfolio with no strategy constraints. We show that when the portfolio strategy is constrained, the VaR-RM is not admissible.

Let $\mathcal{C}$ be the set of all admissible portfolio strategies where short-selling is prohibited and the agent is not allowed to borrow risk-free bonds or cash to finance the purchase of further risky assets:

$$\mathcal{C} = \{\theta^i \geq 0, i = 1, 2, ...d; \sum_1^d \theta^i \leq 1\}.$$

Using the power utility function, we can define the VaR-RM problem in the constrained model as

$$\begin{aligned} & \max_{\theta^{VaR}} E\left[\frac{(W_T^{VaR})^{1-\gamma}}{1-\gamma}\right] \qquad (28) \\ s.t. \quad & P(W_T^{VaR} \geq \underline{W}) \geq 1 - \alpha, \\ & \mathcal{C} = \{(\theta_t^{VaR})^i \geq 0, i = 1, 2, ...d; \sum_1^d (\theta_t^{VaR})^i \leq 1, \forall t \in (0, T)\}, \end{aligned}$$

with a given initial W_0.

Theorem 5. *Let θ_t^{VaR} represent the portfolio strategy of the VAR-RM agent. For any θ_t^{VaR}, the sum exceeds one, i.e.*

$$\sum_{i=1}^d (\theta_t^{VaR})^i \geq 1,$$

with positive probability. Hence the strategy θ_t^{VaR} is not admissible for the problem (28).

5.4 Portfolio insurance with spreads

The Value-at-Risk based risk management was developed for portfolios with no constraints on the portfolio strategy. We showed that such a strategy is useless when constraints, such as restricting the short selling of all risky or risk-free assets, are required. Insuring the portfolio with a put spread can eliminate this problem.

According to the VaR-constraint (26) we adjust our strategy in a following way:

- in case the risky asset X_T satisfies the condition, we do not insure the portfolio at all,
- in case the risky asset X_T does not satisfy the condition, we modify the portfolio by buying a put option with the strike price $\underline{W}$ and selling a put option with strike price $\underline{\underline{W}}$ such that $P(X_T \geq \underline{\underline{W}}) = 1 - \alpha$.

Formally, we can express the above strategy as

$$W = \begin{cases} X + Put(X_T \geq \underline{W}) - Put(X_T \geq \underline{\underline{W}}) & \text{if } P(X_T \geq \underline{W}) < 1 - \alpha, \\ X & \text{if } P(X_T \geq \underline{W}) \geq 1 - \alpha. \end{cases} \tag{29}$$

According to this strategy, we leave the worst $\alpha\%$ cases uninsured.

From Ito's lemma, the condition $P(X_T \geq \underline{W}) \geq 1 - \alpha$ can be expressed as $\underline{W} \leq \underline{\underline{W}}$, where $\underline{\underline{W}} = X_0 e^{\Gamma}$ with

$$\Gamma = \left(r + \beta^{\top}(\mu - r\mathbf{1}) - \frac{1}{2}\sigma_X^2 \right) T - \sigma_X \sqrt{T} \Phi^{-1}(1 - \alpha).$$

In case that $\underline{\underline{W}} < \underline{W}$ and the put options are synthesized, we can express the portfolio value as

$$\begin{aligned} W_t &= X_t + Put_t(X_T \geq \underline{W}) - Put_t(X_T \geq \underline{\underline{W}}) \\ &= X_t + \varphi_t(\underline{W})X_t + \psi_t(\underline{W})B_t - \varphi_t(\underline{\underline{W}})X_t - \psi_t(\underline{\underline{W}})B_t \\ &= \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] X_t + [\psi_t(\underline{W}) - \psi_t(\underline{\underline{W}})]B_t, \end{aligned}$$

where $\varphi_t(\underline{W}) = \Phi(d_1(\underline{W})) - 1$ is the delta of the option with strike $\underline{W}$ and $\varphi_t(\underline{\underline{W}}) = \Phi(d_1(\underline{\underline{W}})) - 1$ is the delta of the option with strike $\underline{\underline{W}}$. The difference $\varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})$ is called the hedging ratio.

Note that is case when $\underline{\underline{W}} \geq \underline{W}$, it holds that $W_t = X_t$.

5.5 Portfolio insurance with spreads in the unconstrained models

We investigate the problem

$$\begin{aligned} & \max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & P(W_T \geq \underline{W}) \geq 1 - \alpha. \end{aligned} \tag{30}$$

In this case, there are no constraints required on the portfolio strategy.

Theorem 6. *Let $\hat{\beta}$ be the optimal portfolio strategy, computed by (8). Then the portfolio strategy defined by*

$$\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases} \tag{31}$$

guarantees that $P(W_T \geq \underline{W}) \geq 1 - \alpha$.

5.6 Portfolio insurance with spreads in the constrained models

We investigate the problem

$$\begin{aligned} & \max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad (32) \\ s.t. \quad & P(W_T \geq \underline{W}) \geq 1 - \alpha, \\ & \mathcal{C} = \{\theta^i \geq 0,\ i = 1, 2, ..., d; \sum \theta^i \leq 1\}. \end{aligned}$$

We provide an admissible solution for the problem (32) in the following theorem.

Theorem 7. *Let $\hat{\beta}$ be the optimal portfolio strategy with convex constraints, computed by*

$$\hat{\beta} = \arg\max_{\beta \in \mathcal{C}} \ \beta^\top(\mu - r\boldsymbol{1}) - \frac{1}{2}\gamma\beta^\top c^R \beta, \tag{33}$$

where $\mathcal{C}$ is defined as in problem (32). Then the portfolio strategy

$$\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases}$$

is admissible for the problem (32).

5.7 Example

Since the VaR-RM is not admissible in the constrained model, we can compare the certainty equivalent of the portfolio insurance with spreads in the constrained model with the certainty equivalent of the VaR-RM in the unconstrained model. We set the risk-free interest rate $r = 2\%$, the parameter of the utility function $\gamma = 5$, the initial wealth $W_0 = 1$, the time to maturity $T = 1$, probability level $\alpha = 0.05$, the expected returns on

$\underline{W}$	C^{VaR}	C
0.98	1.089074	1.070260
0.99	1.088262	1.066877
1	1.087253	1.062447
1.01	1.086015	1.056451
1.015	1.085303	1.052541

Table 2: Certainty equivalents of the VaR-RM and the of the portfolio insurance with spreads in the constrained model.

the risky assets $\mu = (0.06626, 0.1113, 0.1625)$ and the covariance matrix as $c^R = \begin{pmatrix} 0.02155 & 0.00825 & 0.00749 \\ 0.00825 & 0.01517 & 0.01190 \\ 0.00749 & 0.01190 & 0.05011 \end{pmatrix}$, based on data analysis. Table (5.7) compares the certainty equivalents of the VaR-RM and of the portfolio insurance with spreads for different levels of the floor.

The certainty equivalents of the portfolio insurance with spreads are significantly lower than those of the VaR-RM. The average difference between the certainty equivalents of the VaR-RM and of the portfolio insurance with spreads in the constrained model is approximately 2.5%.

6 Conclusions

The main objective of our work was to examine the portfolio insurance when short-selling of both risky and risk-free assets is prohibited. Our goal was to provide a dynamic portfolio strategy that satisfies such constraints and maximizes the expected utility from the partially guaranteed terminal wealth.

Assuming that the terminal wealth of the portfolio is not allowed to fall under the predefined level with probability one and that short-selling is prohibited, we provided two admissible strategies, the OBPI in the constrained model and the alternative method. Based on the results of sensitivity analysis, we concluded that none of the methods dominates the other.

Under the assumption that the terminal wealth is partially allowed to fall under the predefined floor, the Value-at-Risk based risk management in the constrained model turned out not to be admissible, hence we provided an alternative to it, the portfolio insurance with spreads, which is an admissible solution. The portfolio insurance with spreads is not an optimal strategy, hence its certainty equivalents were significantly lower that the certainty equivalents of the VaR-RM.

References

[1] S. Basak and A. Shapiro: Value-at-Risk Based Risk management: Optimal Policies and Asset Prices, Review of Financial Studies 14, 371-405, 2001.

[2] P. Bertrand, J.-L. Prigent: Portfolio Insurance Strategies: Obpi Versus Cppi (December 2001). University of CERGY Working Paper No. 2001-30; GRE-QAM Working Paper. Available at SSRN: `http://ssrn.com/abstract=299688`

[3] N. H. Hakansson: Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions, Econometrica, Vol. 38, No. 5, pp. 587-607, 1970

[4] H.E. Leland and M. Rubinstein: Replicating options with positions in stock and cash, Financial Analysts Journal, Vol. 37, No. 4, pp. 63-71, 1981

[5] Cs. Krommerová: Expected utility maximization with risk management and strategy constraints, Zborník z prvého česko-slovenského workshopu mladých ekonómov, ISBN 978-80-225-3498-7, electronic document, pp. 1-21, 2012

[6] R. Mehra and E. Prescott: The Equity Premium: a Puzzle. Journal of Monetary Economics, vol. 15, pp. 145-161, 1985

[7] R. C. Merton: Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, The Review of Economics and Statistics, Vol. 51, No. 3, pp. 247-257, 1969.

[8] M. Nutz: Power Utility Maximization in Constrained Exponential Lévy Models, Mathematical Finance, Vol. 22, No. 4, pp. 690-709, 2012

[9] M. Nutz: The Bellman Equation for Power Utility Maximization with Semimartingales, Annals of Applied Probability, Vol. 22, No. 1, pp. 363-406, 2012

[10] A. Perold: Constant proportion portfolio insurance, Harward Business School, unpublished manuscript, August 1986

[11] A. Perold and W. F. Sharpe: Dynamic strategies for asset allocation, Financial Analysts Journal, Vol. 44, No. 1, pp 16-27, January-February 1988

[12] J.-L. Prigent: Portfolio Optimization and Performance Analysis, Chapman & Hall/CRC Financial Mathematics Series, 2007

[13] P. A. Samuelson: Lifetime Portfolio Selection By Dynamic Stochastic Programming, The Review of Economics and Statistics, Vol. 51, No. 3, 239 - 246, 1969.

Publikácie autora

- Expected utility maximization with risk management and strategy constraints, Zborník z prvého česko-slovenského workshopu mladých ekonómov [elektronický dokument]. - Bratislava: Katedra hospodárskej politiky, 2012. - ISBN 978-80-225-3498-7. - s. 1-21 [CD ROM]
- Expected utility maximization with risk management and strategy constraints, abstrakt - MMEI 2012: Joint Czech-German-Slovak Conference [elektronický zdroj], 2012 . - s. 29-30 [online]

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