UNIVERZITA KOMENSKÉHO V BRATISLAVE

FAKULTA MATEMATIKY, FYZIKY
A INFORMATIKY

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Autoreferát dizertačnej práce

INDUKOVANÉ PODGRAFY V SILNO REGULÁRNYCH GRAFOCH

na získanie akademického titulu philosophiae doctor
v odbore doktorandského štúdia

9.1.6 Diskrétna matematika

Bratislava 2015

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na Katedre algebry, geometrie a didaktiky matematiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave.

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Obhajoba dizertačnej práce sa koná .... o .... h
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Diskrétna matematika – 9.1.6 Diskrétna matematika

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Induced Subgraphs in Strongly Regular Graphs

1 Introduction

Strongly regular graphs (also referred to as SRGs) [10] are objects standing somewhere between highly symmetric and random graphs. They form an important class of graphs with applications in various fields, such as coding theory theoretical chemistry or group theory.

SRGs appeared for the first time in the work of R. C. Bose [3], who in cooperation with D. M. Mesner explained their connection to linear algebra [4]. An SRG is a regular graph which can be described by four parameters: its number of vertices (n), its degree (k), the number of common neighbor for any pair of adjacent vertices (λ) and the number of common neighbor for any pair of non-adjacent vertices (μ).

The strongest techniques in this area are provided by spectral graph theory. This fact is illustrated by the influential paper of J. Hoffman and R. Singleton [17] where authors show that there are only four feasible values for the degree of Moore graphs with diameter two (that is SRGs with $\lambda = 0$ and $\mu = 1$), namely 2, 3, 7 and 57. Hoffman an Singleton have successfully constructed three of those, but the existence of the Moore graph of valency 57 remains unknown. Actually, this question became a famous open problem in graph theory. For illustration, we refer to a monograph of A. E. Brouwer and W. H. Haemers [6].

Spectral methods provide extremely strong necessary conditions on the parameters of an SRG. As a consequence, researchers usually apply combinatorial techniques on very limited systems of parameters sets. A notable exception is an important connection between SRGs and design theory [4], [3].

A useful combinatorial technique is the study of interplay of some small configurations in a given object and expressing the number of their occurrences can help derive new conditions for existence of the object. This method is used in [7] where 4-cycle systems are explored.

2 Goals and results

We focus on theoretical and algorithmic tools for determining the numbers of induced subgraphs in SRGs from their parameters and on further applications of such numbers. Although the information about small induced subgraphs was efficiently used in the study of graphs with concrete parameter sets, for example in [6] and [2], we are not aware of results in this direction. Similar methods were used in the study of graphs satisfying so called t-vertex condition.

For a given order o, we develop an algorithm that produces relations between numbers of induced subgraphs of orders o and $o-1$ for an arbitrary SRG. Moreover, the algorithm can be easily modified to consider only triangle-free SRGs or Moore graphs. The relationships are given by a system of linear equations.

Our main result is an extensive analysis of the solutions of these systems. For instance, it turns out that in a putative Moore graph Γ of valency 57, the number of induced subgraphs isomorphic to a given graph on 10 vertices depends only on the number of induced subgraphs isomorphic to the Petersen graph in Γ. Among the applications of our methods, there are new results about automorphisms of order 7 in a Moore graph of valency 57 and new bounds on the numbers of induced $K_{3,3}$ in triangle-free SRGs with small number of vertices.

3 Strongly regular graphs - Basic definitions and known results

Definition 3.1. Strongly regular graph *with parameters* (n, k, λ, μ) *is a k-regular graph on n vertices with following properties:*

1. *Any two adjacent vertices have exactly* λ *common neighbours.*
2. *Any two non-adjacent vertices have exactly* μ *common neighbours.*

Note that any disconnected SRG has to be a disjoint union of complete graphs of the same order and it is considered to be of little interest. Since the complementary graph of any SRG is also strongly regular we consider only SRGs for which both their complements and themselves are connected. SRGs of this kind are called primitive. If some SRG is not primitive we call it imprimitive. Note that SRGs with disconnected complements are complete multi-partite graphs.

Let A be an adjacency matrix of some graph Γ. We can observe that Γ is an SRG with parameters n, k, λ and μ if and only if A satisfies

i) A has row sum k

ii) If J denotes the all ones matrix and I the identity matrix, then

$$A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J \qquad (1)$$

Theorem 3.1. *The spectrum of any $SRG(n, k, \lambda, \mu)$ consists of eigenvalues k, r and s, where*

$$r, s = \frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

Their respective multiplicities are 1, f and g with f and g satisfying

$$f, g = \frac{1}{2}\left(n - 1 \mp \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}}\right)$$

Let n, k, λ and μ be non-negative integers. Then $SRG(n, k, \lambda, \mu)$ exists only if the expression

$$\frac{(n-1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}}$$

is an integer with the same parity as $n - 1$. The parameter sets that can be obtained from this criterion are called feasible.
On the basis of this result we can classify SRGs into two classes. A table of feasible values of parameters for SRGs on up to 1300 vertices can be found on the home page of A. E. Brouwer [5].

3.1 Krein condition

Proposition 3.1 (Krein condition)**.** *[18] Let Γ be an arbitrary $srg(n, k, \lambda, \mu)$ with eigenvalues k, r and s. Then the following holds*

$$(r + 1)(k + r + 2rs) \leq (k + r)(s + 1)^2$$
$$(s + 1)(k + s + 2rs) \leq (k + s)(r + 1)^2$$

An *SRG* is called a *Krein graph* if an only if the equality holds in one of Krein's conditions. Let us fix a vertex v in some Krein graph. Then the subgraphs induced by neighbours of v and by its non-neighbours are both strongly regular. These are called *linked graphs* [11].

Lemma 3.1. *[12] Let* Γ *be a triangle free Krein graph with* $2 < \mu < k$. *Then,* Γ *has a parameter set* $((r^2+3r)^2, r^3+3r^2+r, 0, r^2+r)$ *iff any triple of vertices inducing a* $\overline{K}_3$ *in* Γ *has exactly* r *common neighbours.*

Proposition 3.2. *[15]*

- *There is no* $Kr(3)$.
- $Kr(r)$ *does not contain* $K_{r,r}$ *as an induced subgraph for* $r \geq 9$

3.2 absolute bound and geometry of the spectra

Let Γ be an $srg(v, k, \lambda, \mu)$ and let A be its adjacency matrix. We will use the notation l for the number of vertices that are not adjacent to a given vertex. As it was already shown in the theorem 3.1 the matrix A has exactly three eigenvalues (k of multiplicity 1, r of multiplicity f, s of multiplicity g).

Any vertex v_i of Γ can be represented by the i^{th} row of the adjacency matrix A. As we consider only primitive SRGs, all eigenvalues of A are non-zero. It follows that the set of vertices represented by vectors $\{v_1, \ldots, v_n\}$ forms a basis of $\mathbb{R}$. Let V_0, V_1, V_2 denote eigenspaces for eigenvalues k, r, s respectively and let E_1, E_2 and E_3 be orthogonal projection onto the eigenspace V_i. Hence the adjacency matrix A can be rewritten as the sum

$$A = kE_0 + rE_1 + sE_2.$$

Now, let us restrict our attention to one of the non-trivial eigenspaces V_2. We will consider here (normalized) projections of "vertices" v_i into this eigenspace:

$$x_i = \frac{v_i E_2}{||v_i E_2||}$$

It is easy to verify the following property of these vectors.

Proposition 3.3. *Let* Γ *be a primitive* $srg(n, k, \lambda, \mu)$ *wit eigenvalues* $k > r > s$ *and let* x_i *for* $i \in \{1, \ldots, n\}$ *denote orthogonal projection of the vertex* v_i *onto eigenspace* *corresponding to the eigenvalue s. Then the inner product of two vectors x_i and x_j satisfies:*

$$x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j, \\ p & \text{if } i \sim j, \\ q & \text{otherwise,} \end{cases}$$

where $p = \frac{s}{k}$ and $q = -\frac{(s+1)}{(n-k-1)}$.
Moreover, as Γ is connected and not complete multipartite, $x_i \neq x_j$ for $i \neq j$.

The proposition above says that all created vectors lie on a unit sphere in g-dimensional Euclidean vector space with a specific distance across each other. Delsarte et al. provide the following result:

Theorem 3.2 (Absolute bound [14])**.** *Let $g \neq 1$ be a multiplicity of an eigenvalue of a $srg(n, k, \lambda, \mu)$. Then*

$$n \leq \binom{g+2}{2} - 1$$

3.3 Triangle-free SRGs

There are only seven known examples of tfSRGs. Three of them are members of the famous family of Moore graphs with diameter two. Their sets of parameters are $(5, 2, 0, 1)$ (pentagon), $(10, 3, 0, 1)$ (Petersen graph) and $(50, 7, 0, 1)$ (Hoffman-Singleton graph), $(16, 5, 0, 2)$ (Clebsch graph), $(56, 10, 0, 2)$ (Sims-Gewirtz graph), $(77, 16, 0, 4)$ (Mesners M_{22} graph) and $(100, 22, 0, 6)$ (Mesner, Higman-Sims graph). Each of these graphs is uniquely determined by its parameters and it is unknown whether there are any further graphs.

Note that all known primitive $tfSRGs$ can be found as induced subgraphs of some larger one [21]. Known results are presented in the table 1. The remarkable fact is that any known tfSRG does not contains $K_{3,3}$ as induced subgraph. On the other, each $Kr(r)$ with $r > 2$ has to contain induced $K_{3,3}$. According to lemma 3.1, their number is determined uniquely. Hence, the questions about induced $K_{3,3}$ in $tfSRGs$ are natural.
Obviously, the missing Moore graph does not contain induced $K_{3,3}$. Yet there is another interesting open problem. It is not known whether it can contain induced Petersen graph.

	Pentagon	Petersen	Clebsch	HoSi	SimGe	Mesner	HiSim
Pentagon	1	12	192	1260	8060	88704	443520
Petersen		1	16	525	13440	1921920	35481600
Clebsch			1	0	0	0	924000
HoSi				1	0	0	704
SimGe					1	22	1030
Mesner						1	100
HiSim							1

Table 1: Number of tfSRGs inside of tfSRGs [21]

4 Small subgraphs in SRGs

Let Γ denote some $srg(n, k, \lambda, \mu)$. For every graph G on t vertices there is a value representing the number of occurrences of G as an induced subgraph in Γ. This value is for some G constant and depends only on parameters n, k, λ and μ. For example, it is easy to derive a number of triangles in Γ.

4.1 Equations for a general SRG

Suppose that numbers of all subgraps on $t-1$ vertices are given. We use strong regularity to construct special equations for each of $(t-1)$-vertex graph.
Let G be a graph on $t-1$ vertices.

1. We have one equation saying which types of subgraphs can be created by adding one arbitrary vertex from Γ into subgraphs isomorphic to G. The left side of this equation contains the number of occurrences of G as induced subgraph in Γ multiplied by $n-(t-1)$. The right side consists of the sum of occurrences of all t-vertex graphs as induced subgraphs in Γ. Each of summands is multiplied by a constant saying how many times G occurs in the appropriate graph.
2. For every vertex v of G there is an equation describing subgraphs, which we can obtain by adding a vertex adjacent to v.
3. For any pair of adjacent vertices x and y in G we have an equation for subgraphs which can be created by adding a common neighbor of x and y.
4. For any pair of non-adjacent vertices x, y in G there is an equation for subgraphs which can be created by adding their common neighbors.

4.2 The algorithm

Definition 4.1. *Let us fix a graph G. Then the function $P_G(H)$ computes the number of occurrences of G in a graph H as induced subgraph.*

The input:

- Parameters of the graph $\Gamma = srg(n, k, \lambda, \mu)$
- The list $L_0 = \{G_1, G_2, \dots, G_{|L_0|}\}$ of all graphs of order $o - 1$
- The values $P_G(\Gamma)$ for every graph $G \in L_0$
- The list L of graphs of order o

The output:

- System of linear equations, which are satisfied by values $P_H(\Gamma)$, for $H \in L$

The procedure:

(1) Initialize i=1

(2) Take $G_i \in L_0$

(3) Find all H of order o containing G_i as induced subgraph

(4) Determine the equation that follows from parameter n of Γ

(5) Determine all equations that follow from parameter k of Γ

(6) Determine all equations that follow from parameter λ of Γ

(7) Determine all equations that follow from parameter μ of Γ

(8) Increase i by 1.

(9) If $i \leq |L_0|$, return to step (2)

(10) Return:

The system of linear equations in the form $Mx = b$ satisfied by $P_H(\Gamma)$, for $H \in L$.

The algorithm works with parameters n, k, λ and μ as with abstract variables, therefore the solution is universal for any SRG. However, the matrix M contains just integers and does not depend on the choice of SRG. Since the right hand side of the output is derived using n, k, λ and μ, the solution (possible values of $P_H(\Gamma)$, for $H \in L$) can be also expressed by these parameters.
Steps 4, 5, 6, 7 are provided using statements in the following proposition:

Proposition 4.1. *Let L be the set of all graphs on o vertices. For $H \in L$, the vertex u_j will be a representative of the orbit V_H^j. If G is a graph of order $o-1$ then the following statements holds:*

i.

$$(n-o+1)P_G(\Gamma) = \sum_{H\in L} P_G(H)P_H(\Gamma)$$

ii. For any $v \in V_G^i$ with $deg_G(v) \leq k$

$$(k - deg_G(v))|V_G^i|P_G(\Gamma) = \sum_{H\in L}\sum_{j} f(V_G^i, H, u_j)|V_H^j|P_H(\Gamma),$$

iii. For any edge $(v_1, v_2) \in E_i^G$ such that $\deg_G(v_1, v_2) \leq \lambda$

$$(\lambda - deg_G(v_1, v_2))|E_G^i|P_G(\Gamma) = \sum_{H\in L}\sum_{j} f(E_i^G, H, u_j)|V_H^j|P_H(\Gamma),$$

iv. For any non-edge $(v_1, v_2) \in \overline{E}_G^i$ such that $\deg_G(v_1, v_2) \leq \mu$

$$(\mu - deg_G(v_1, v_2))|\overline{E}_G^i|P_G(\Gamma) = \sum_{H\in L}\sum_{j} f(\overline{E}_G^i, H, u_j)|V_H^j|P_H(\Gamma).$$

4.3 Results

The solution of the system of equations described in the previous part gives us total numbers of all induced subgraphs of a given order in some $srg(n, k, \lambda, \mu)$. This solution can be for some *SRGs* unique (iff the rank of the system is equal to the number of created graphs) but in general it is not. Hence we are obtaining a set of solutions with the integrability requirement. In this case, there are dependences between numbers of occurrences of subgraphs.

Theorem 4.1. *Let* Γ *always denotes* $srg(n, k, 0, \mu)$.

- *The value* P_G *is for any graph on at most* 5 *vertices determined uniquely by parameters* k *an* μ.
- *There are* 12 *graphs on* 6 *vertices for which their numbers of occurrences as induced subgraphs in* Γ *are determined uniquely only by parameters* k *and* μ. *The values* P_G *for remaining* 26 *cases depend also on the number of induced* $K_{3,3}$ *subgraphs in* Γ, *which is represented by parameter* P_1.
- *There are* 15 *graphs on* 7 *vertices for which their numbers of occurrences in* Γ *are determined uniquely by values* k *and* μ. *Between remaining* 92 *cases there are* 91 *graphs whose occurrences depend also on* $P_{K_{3,3}} =: P_1$ *and occurrences of* 76 *of them (including* $K_{3,4}$*) depend on* $P_{K_{3,4}} =: P_2$.

Theorem 4.2. *Let* Γ *always denotes* $srg(3250, 57, 0, 1)$. *Then number of occurrences of any graph on at most* 9 *vertices in* Γ *is constant.*
There are 595 *graphs of order* 10 *whose number of occurrences in* Γ *is constant. Occurrences of remaining* 274 *cases depend on the number of induced Petersen graphs in* Γ. *The upper bound for the number of induced Petersen graph is* 266 266 000.

Theorem 4.3. *For any graph* G *of order* 4 *the value* P_G *for* $\Gamma = srg(n, k, \lambda, \mu)$ *depends on parameters* k, λ, μ *and on the value* $P_1 := P_{K_{1,3}}$.

5 Geometry of the spectra of srg

This chapter brings two approaches for dealing with SRGs together. The first comes from the algorithm that we invented. The second follows from the geometrical representation of SRGs. The idea comes from the work of Bondarenko, Prymak and Radchenko [19].

5.1 Zonal spherical harmonics

This section describes an alternative view on the geometrical representation of SRG [19]. Instead of working directly with vectors of sphere S^{g-1} we will translate the whole problem into the world of special polynomials with inputs from S^{g-1}. For more details we recommend [13] and [16].

Theorem 5.1 (Riesz representation theorem)**.** *Let* H *be a Hilbert space with the inner product* $\langle .,. \rangle$ *and let* H^* *be the dual space of* H. *For any functional* $g \in H^*$ *there exists unique* $y \in H$ *such that*

$$g(x) = \langle x, y \rangle, \ \forall x \in H$$

Definition 5.1. *For a fixed vector $x \in S^{g-1}$, the zonal spherical harmonic Z_x^t of degree t is defined as the dual (Riesz) representation of the mapping $P \mapsto P(x)$ where $P \in \mathcal{H}_t(S^{g-1})$. In other words $Z_x^t(y)$ satisfies the following reproducing property*

$$P(x) = \int_{S^{g-1}} Z_x^t(y) P(y) d\omega_g(y),$$

for all $P \in \mathcal{H}_t(S^{g-1})$.

The following property follows from the uniqueness of the element y in Theorem 5.1.

Lemma 5.1. *The zonal spherical harmonics satisfy the following equality.*

$$\left\langle Z_y^t, Z_x^t \right\rangle = \int_{S^{g-1}} Z_y^t(\xi) Z_x^t(\xi) d\omega_g(\xi) = Z_y^t(x)$$

Each polynomial Z_x^t is an element of $\mathcal{H}_t(S^{g-1})$, which is a Hilbert space. Hence we can formulate the Cauchy-Schwartz inequality for Z_x^t and Z_y^t.

$$\langle Z_x^t, Z_y^t \rangle^2 \leq \langle Z_x^t, Z_x^t \rangle \langle Z_y^t, Z_y^t \rangle$$

Using lemma 5.1, which uniquely gives the value of the inner product of two zonal harmonics , we have:

$$\left(\sum_{i,j} Z_{x_i}^t(y_j) \right)^2 \leq \sum_{i,i'} Z_{x_i}^t(x_{i'}) \sum_{j,j'} Z_{y_j}^t(y_{j'}) \qquad (2)$$

For our purpose the definition of zonal spherical harmonics that use Gegenbauer polynomial is more useful.

Definition 5.2. *The Gegenbauer polynomial $C_t^\alpha(x)$ is defined recursively as*

$$\begin{aligned} &C_0^\alpha(x) = 1, \\ &C_1^\alpha(x) = 2\alpha x, \\ &\qquad \vdots \\ &C_t^\alpha(x) = \frac{1}{t} \left[2x(\alpha + t - 1) C_{t-1}^\alpha(x) - (2\alpha + t - 2) C_{n-2}^\alpha(x) \right]. \end{aligned}$$

Proposition 5.1. *The zonal spherical harmonic $Z_x^t(y)$ can be expressed as follows.*

$$Z_x^t(y) = \frac{1}{c_{g,t}} C_t^\alpha(x \cdot y),$$

where $\alpha = \frac{g-2}{2}$ and $x \cdot y$ is the usual inner product of vectors $x, y \in \mathbb{R}^g$. The value $c_{g,t}$ is constant and satisfies

$$c_{g,t} = \frac{1}{\omega_{g-1}} \frac{2t+g-2}{g-2},$$

where ω_{g-1} denotes the surface area of the $g-1$ dimensional sphere.

After the transition to new definition of zonal harmonics we obtain an easy way to compute their values for given vectors x and y. Moreover, the constant $c_{g,t}$ does not depends on the choice of x and y. It follows that the inequality 2 can be rewritten in a more usable form.

Proposition 5.2. *Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be finite subsets of points of the unit sphere S^{g-1}. Then the Gegenbauer polynomial C_t^α with $\alpha = \frac{g-2}{2}$ has the following property*

$$\left(\sum_{i,j} C_t^\alpha(x_i \cdot y_j) \right)^2 \leq \sum_{i,i'} C_t^\alpha(x_i \cdot x_{i'}) \sum_{j,j'} C_t^\alpha(y_j \cdot y_{j'})$$

5.2 Gegenbauer polynomials and induced subgraphs in SRG

Let us recall the geometrical representation of $\Gamma = srg(n, k, \lambda, \mu)$ from the section 3.2 constructed by the spectra of Γ.

Let Δ be a subgraph of Γ induced by vertices $\{v_1^\Delta, v_2^\Delta, \dots, v_o^\Delta\}$. The vector x^Δ constructed as

$$x^\Delta = \frac{\sum_{i=1}^o x_i^\Delta}{|| \sum_{i=1}^o x_i^\Delta ||}$$

will represent the subgraph Δ on the sphere S^{g-1}.

Lemma 5.2. *Let us continue with notation L, X and Y in correspondence with the discussion above. Then*

$$\sum_{\substack{\Delta\in X\\ \Omega\in Y}} C(x^{\Delta}\cdot x^{\Omega})$$
$$\|$$
$$\sum_{W\in L} P_W(\Gamma) \sum_{\substack{\Delta,\Omega\subset W\\ V_\Delta\cup V_\Omega=V_W}} \frac{|V_\Delta\cap V_\Omega| + e(\Delta,\Omega)p + \overline{e}(\Delta,\Omega)q}{\sqrt{\left(|V_\Delta| + 2p|E_\Delta| + 2q|\overline{E}_\Delta|\right)\left(|V_\Omega| + 2p|E_\Omega| + 2q|\overline{E}_\Omega|\right)}},$$

where Δ and Ω represent fixed subgraphs in Γ or in W that are isomorphic to G and H, respectively. $P_W(\Gamma)$ is the number of induced subgraphs of Γ that are isomorphic to W.

5.3 Results

Lemma 5.3. *Let us consider Krein graph $Kr(r)$. The number of induced $K_{3,3}$ in this SRG is equal to $\frac{1}{2}P_{\overline{K_3}}\binom{r}{3}$*

All results for tfSRGs are summarized in Table 2. The first column of the table represents parameter set of the considered tfSRG. The second column belongs to the upper bound coming from the solution of the system of linear equations obtained by our algorithm for triangle-free case. The heading row describes the choice of X, Y and t. The illustration of the results for even t can be found in the last column, where the lower bound is negative.
First four parameter sets represented in the table are Krein graphs. We can see that the results obtained geometrical approach are better for those graphs, where the Krein inequality is tighter.

We have tried to obtain new bounds also for the number of induced Petersen graphs in the missing Moore graph by this method. The upper bound following from our algorithm equals 266 266 000 and the lower bound is 0. The bounds obtained from the geometrical approach are for all cases which we tested much worse.

tfSRG n	k	μ	lin. equations	X K_1, Y $K_{1,2}$, t 5	X $K_{1,1}$, Y $K_{1,2}$, t 5	X $K_{1,1}$, Y $K_{1,2}$, t 6
100	22	6	51 333.33	0.00	0.00	-2 189.08
324	57	12	15 800 400.00	1 580 040.00	1 580 040.00	-717 648.25
784	116	20	894 206 880.00	99 356 320.00	99 356 320.00	-171 378 969.69
1600	205	30	21 129 322 667.00	2 263 856 000.00	2 263 856 000.00	-384 925 3316.17
77	16	4	3 080.00	534.63	552.27	-378.11
162	21	3	1 890.00	124 452.56	42 409.60	-36 672.31
176	25	4	17 600.00	218 854.15	74 550.28	-60 480.70
210	33	6	246 400.00	524 614.17	210 718.23	-136 110.87
266	45	9	2 867 480.00	1 186 988.94	730 655.25	-350 790.83
352	36	4	73 920.00	20 793 353.15	1 851 582.50	-2 252 418.82
392	46	6	901 600.00	46 206 927.87	3 943 905.78	-4 572 373.08
552	76	12	48 070 000.00	202 641 567.42	24 938 590.19	-25 958 718.19
638	49	4	250 096.00	546 336 456.15	17 861 596.71	-35 169 414.63
650	55	5	965 250.00	78 212 8175.56	22 335 864.64	-46 682 434.69
667	96	16	248 390 800.00	286 405 500.67	57 611 096.19	-70 157 307.76
800	85	10	45 696 000.00	3 196 736 243.94	70 873 609.74	-153 636 540.68
1073	64	4	721 056.00	7 483 465 849.76	128 237 091.68	-331 449 507.48

Table 2: Bounds for induced $K_{3,3}$ in $tfSRGs$ on up to 1100 vertices

6 Automorphism group of the missing Moore graph

The automorphism group of $\Gamma = srg(3250, 57, 0, 1)$ is well studied. In 1971 Aschbacher proved that Γ is not rank three graph [1] and Higman showed in his unpublished notes that Γ cannot be vertex transitive (for the proof see Cameron monograph [9]). Later in 2001, Makhnev and Paduchikh showed that if Γ has an involutive automorphism then $|Aut(\Gamma)| \leq 550$ [20]. Finally in 2009 Mačaj and Širáň proved that if $|Aut(\Gamma)|$ is odd then $|Aut(\Gamma)| \leq 275$ and $|Aut(\Gamma)| \leq 110$ otherwise [8].

Proposition 6.1. *[8] Let x be an automorphism of graph Γ of order 7. Then the value $a_1(x)$ satisfies*

$a_0(x)$	p	$a_1(x)$
2	7	$49 + 105k \leq 500$
9	7	$98 + 105k \leq 500$
16	7	$42 + 105k \leq 500$
23	7	$91 + 105k \leq 500$
30	7	$35 + 105k \leq 500$
37	7	$84 + 105k \leq 392$
44	7	$28 + 105k \leq 260$
51	7	77

We apply our results about small subgraphs in tfSRG to give a new information about automorphism of Γ of order 7.

Proposition 6.2. *The number of occurrences of any 7-vertex graph except $\overline{K_7}$ in Γ as induced subgraph is a multiple of 7. The number of induced $\overline{K_7}$ in Γ is equal to* 2 mod 7.

Theorem 6.1. *Let x be an automorphism of $srg(3250, 57, 0, 1)$ of order 7. Then the value $a_1(x)$ is a multiple of 49.*

$a_0(x)$	$a_1(x)$
2	49
9	98
16	147
23	196
30	245
37	294
44	–
51	–

Proposition 6.3. *The number of occurrences of any graph G of order 10 in $srg(3250, 57, 0, 1)$ with no induced Petersen is congruent to* $0 \mod p$ *for each* $p \in \{7, 11, 13, 19\}$.

7 Conclusion

Strongly regular graphs, SRGs, are characterized by two simple conditions. They are regular and the number of common neighbors of a pair of vertices in a given SRG depends only on whether they are adjacent or not. In spite of the simplicity of this characterization there are significant applications of SRGs in combinatorics, algebra, statistics, and group theory, including discovery of several sporadic simple groups.

The goal of this thesis was to find theoretical and algorithmic tools for determining the numbers of induced subgraphs in strongly regular graphs just from their parameters n, k, λ, μ and further applications of such numbers. We considered in more detail a restricted class of these graphs, specifically those with no triangles. In this special case, there are infinitely many feasible sets of parameters for SRGs. Despite this fact there are only seven known examples of such graphs.

One of the most important parts of our work is an algorithm which produces linear equations describing various relations between numbers of induced subgraphs of orders

o and $o-1$ in an SRG. The analysis of the systems of equations provided by our algorithm gives rise to the main result of the thesis. Namely, we are able to prove that in any $srg(n,k,0,\mu)$ the number of induced subgraphs isomorphic to a given graph G on up to 7 vertices depends only on parameters k and μ and on the number of induced $K_{3,3}$s and $K_{3,4}$s. Moreover, in any $srg(3250,57,0,1)$, the number of induced subgraphs isomorphic to a given graph on up to 10 vertices depends only on the number of induced Petersen graphs in this SRG.

Among the applications of our methods, there were new results about automorphisms of $srg(3250,57,0,1)$ and new bounds for the numbers of induced $K_{3,3}$ in triangle-free SRGs. At the end of the thesis we discussed possible extension of our approach for the study of t-vertex condition.

Despite our best effort we were not able to show that Moore graph of valency 57 has to contain a Petersen graph as induced subgraph. We feel that it is worthwhile to look for Petersen-free Moore graphs.

As it was observed in Section 5 in the case of small triangle free Krein graphs, our computations provide tight bounds for induced $K_{3,3}$ subgraphs in these SRGs. It would be nice to have the confirmation of this observation for all triangle-free Krein graphs.

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