Univerzita Komenského v Bratislave

Fakulta matematiky, fyziky a informatiky

Samuel Kováčik

Autoreferát dizertačnej práce

Aspects of physics in noncommutative space

na získanie akademického titulu philosophiae doctor

v odbore doktorandského štúdia:

4.1.2. Všeobecná fyzika a matematický fyzika

Bratislava, 27.4.2016

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na Katedre teoretickej fyziky a didaktiky fyziky, FMFI UK

Predkladateľ: Mgr. Samuel Kováčik
KTFDF FMFI UK
Mlynská dolina
842 48 Bratislava

Školiteľ: Prof. Peter Prešnajder
KTFDF FMFI UK
Mlynská dolina
842 48 Bratislava

Oponenti: ……………………………….
……………………………….
……………………………….
……………………………….
……………………………….
……………………………….
……………………………….

Obhajoba dizertačnej práce sa koná …… o ……. h pred komisou pre obhajobu dizertačnej práce v odbore doktorand-ského štúdia vymenovanou predsedom odborovej komisie ……………………………. .

4.1.2. Všeobecná fyzika a matematická fyzika

na ……………………………………………

Predseda odborovej komisie:
....
....
....

Abstract:

In the presented work we analyze a model of quantum mechanics (QM) in a three dimensional rotationally invariant noncommutative (NC) space. Signature features of the models are impossibility of measuring exact positions and absence of infinitely large energies (which are dual). The key tool for the study is the velocity operator, which in the ordinary QM equal to the gradient or momentum operator. Analysis of this operator, together with its relation to the other significant operators of the theory, reveals a (surprising) SO(4)$\rhd$T(4) structure. A dynamical problem is analyzed as well, probably the most important one in QM - the hydrogen atom. The problem is solved in an algebraic way in means of the Laplace-Runge-Lenz vector, with a method developed by Pauli. The results are in perfect accordance with those obtain by direct differential methods carried out earlier by Prešnajder and Gáliková. It is shown that a certain generalization of the model leads to an inclusion of magnetic monopoles, which is proved by comparing both kinematical and dynamical structure of the theory with that of monopoles. (Semi)classical physical consequences for two phenomena sensible to UV cutoff are investigated: radiation of microscopic black holes and Casimir effect on a line. In the former one space noncommutativity leads to replacing an extremely hot stage of a microscopic black hole with a frozen one, in the latter one to results similar to those of linear confinement potential with correction terms.

Keywords: Quantum theory, noncommutative geometry

Abstrakt:

V predkladanej práci prezentujeme analýzu modelu kvantovej mechaniky (QM) v trojrozmernom rotačne invariantnom nekomutatívnom (NC) priestore. Znakmi tohoto modelu sú nemožnosť presnej lokalizácie polohy a absencia nekonečne veľkých energií (tieto koncepty sú duálne). Kľúčovým nástrojom štúdie bol operátor rýchlosti, ktorý je v bežnej QM ekvivalentný operátoru gradientu alebo hybnosti. Analýza tohoto operátora a jeho vzťahov s ďalšími dôležitými operátormi v teórii sa odhaľuje (prekvapivá) SO(4)$\rhd$T(4) štruktúra. Dynamický problém analyzujeme tiež, pravdepodobne ten najdôležitejší v QM – atóm vodíka. Úloha je riešená algebraicky pomocou Laplace-Runge-Lenzovho vektora, metódou objavenou Paulim. Výsledky sú v presnej zhode s tými, ktoré boli získané priamou diferenciálnou metódou Prešnajderom a Gálikovou. Ukázali sme, že isté zovšeobecnenie modelu odpovedá zahrnutiu magnetických monopólov, čo sme dokázali porovnaním kinematických aj dynamických vzťahov v teóriach. Boli preskúmané (semi)klasické fyzikálne dôsledky dvoch javov citlivých na UV cutoff: žiarenie mikroskopických čiernych dier a Casimirov jav na priamke. V prvom znich viedla nekomutatívnosť priestoru k nahradenie extrémne horúceho štádia mikroskopických čiernych dier zamrznutým, v druhom prípade viedla k výsledkom podobným pre potenciál uväznenie (kvarkov) s korekčnými členmi.

Kľúčové slová: Kvantová teória, nekomutatívna geometria

Aspects of physics in noncommutative space

In the presented work we have analyzed a model of quantum mechanics (QM) in noncommutative (NC) space. Contrary to ordinary QM the usual notion of space is abandoned and replaced with one containing small scale structure. In practice (broadly speaking) this means that one cannot distinguish two points of space, if their distance would be smaller than a certain fundamental length, perhaps of the order of Planck length, as an artefact of quantum gravity.

Motivation for NC spaces is twofold. Firstly, thought experiments trying to combine quantum theory with that of gravity suggest that space localization is forbidden under certain range - probing particles become hidden under their event horizons, (microscopic) black holes are formed a no information can be obtained. Obviously this example only shows something peculiar is going on, a proper theory of quantum gravity, which is yet to be discovered, is to be used to investigate the issue. However, the theories that are the best candidates for the quantum gravity - the string theory and its expansion M-theory both contain NC spaces, so it is safe to assume that they will play an important role in the upcoming research.

The other motivation for NC models is the following. Many physical theories are overridden by infinities, which prevent direct calculations and hide the results in a fog of divergencies. If a theory is absent of arbitrarily short distances, it is also absent of arbitrarily large energies, as these are dual. A signature feature of NC models is a natural UV cutoff. This is related to the previous part of the motivation - quantum gravity is nonrenormalizable and in a great need for regularization.

We have decided to study a three dimensional rotationally invariant NC space, for obvious reasons. This model was developed in [3]. Another decision was to study QM instead of quantum field theory (QFT), the other seems to be very difficult to achieve at the moment and it is perhaps better to obtain a good understanding of NC QM and generalize it to NC QFT only afterwards, yet our work aimed to fulfil the first part of this goal.

So far this was the background of our work we have built it upon it. The first step is to define QM. We have constructed the NC space with two sets of auxiliary bosonic creation (c/a) and annihilation operators and used the same for construction of QM. The Hilbert space of states is spanned on monomials in c/a operators (acting in an auxiliary Fock space) and equipped with a Hilbert-Schmidt (trace) norm. Physical operators, most importantly the free Hamiltonian, the position and angular momentum operators are defined using c/a operators and (their) commutators.

The second step is to investigate and analyze the constructed NC QM. The main tool for this was the velocity operator, which is, up to multiplicative constant, equal to the gradient or momentum operator. We have used this operator (and its relation to others) to study the kinematical structure of the theory. And important structure of semi-direct product of SO(4) and T(4) has been revealed. Eigenvalues of a T(4) four-vector (Euclidean, not a relativistic one) composed of three velocity components and a free kinetic energy (a linear function of it) has a crucial property - its amplitude is equal to the inverse of square of the constant defining the noncommutativity of the space. It this is nonzero, obtaining infinite energies is impossible. This exposes the UV regularization of our model of NC QM.

NC QM we have constructed and analyzed its internal structure is a physical theory. Physical theories are tools for describing the universe, defining and solving problems in it. A tool has been constructed, and it should be also demonstrated how to use it properly. The central problem of QM is the hydrogen atom, which is usually solved using appropriate differential (Schrödinger) equation. This method was generalized to NC QM in [3]. The hydrogen atom problem can be also solved algebraically, as was shown by Pauli, with the use of so called Laplace-Runge-Lenz vector. We have generalizes this method to NC QM, the results are in a perfect accordance with those of differential approach.

We have shown that certain generalization of physical states the Hilbert space consists is equal to including magnetic monopoles into the theory. This was proved by comparing both kinematic algebraic structure of the theory, as well as dynamical structure (with Coulomb potential), to the results from the theory of magnetic monopoles. The results are entirely, with a possible correction which vanish in a commutative limit. A new viewpoint on the Dirac quantization condition is offered.

Problem of NC QM is that its corrections are expected to become relevant on the energy scale of Planck energy, which is far above the scope of QM. Therefore we have analyzed two problems semiclassically. We took a result from NC QM and implemented it into a (commutative) classical theory. The first investigated were microscopic black holes, whose internal singularity has been replaced with a blurred (regular) one. Ordinary microscopic black holes evaporate very rapidly and eventually reach infinite temperature. This stage is in NC-inspired approach replaced by a frozen one (this approach was first studied in [7]. Another problem studied semiclassically was a 1 dimensional Casimir effect. The results show strong resemblance with the theory of (linear) quark potential with correction with Lüscher correction terms.

Summing this all up we aimed to obtain a very good and deep understanding of the model of NC QM, show how to define and solved problems within it, analyze the internal structures and find some possible physical consequences. All this shall allow a (future) construction of NC QFT. Let us now comment on all individual parts of the work.

Velocity operator

Goal: Study kinematic relations of the theory with use of the velocity operator, special emphasis is on energy regularization of the theory.

Method: Velocity operator is defined by Heisenberg relation with use of (free) Hamiltonian and coordinate operator(s), which are in turn both expressed in terms of c/a operators. In fact all of the relevant physical operators can be expressed as quadratic combinations of left and right multiplications with c/ a operators. To analyze structure of the theory commutators of all relevant operators (Hamiltonian, velocity, angular momentum and coordinate operators) were computed, together with the relation between square of the velocity operator and the free Hamiltonian. In addition to that velocity eigenstates shall be searched for.

Results: We have discovered that angular momentum and coordinate operators form a representation of so(4) algebra (each alone forms so(3)). Heisenberg uncertainty relation (commutator of velocity operators with coordinate operators) obtains NC correction (compared to ordinary QM), but it is proportional to the Hamiltonian, so does not spoil the algebraic structure. Another surprising result was a vanishing commutator of velocities. Possibly the most important result from this section is the relation between the Hamiltonian and the velocity operator. It has been shown that they together form a SO(4) vector whose amplitude is inversely proportional to the NC constant, resulting into UV regularisation on this scale. Summing this up we have found $\mathrm{SO}(4)\triangleright\mathrm{T}(4)$ structure. Also velocity operators have been found.

Hydrogen atom

Goal: To find the hydrogen atom energy spectrum algebraically and compare the results with those of differential approach.

Method: Generalize the method developed by Pauli in [9] to NC QM. The central object of this approach is the Laplace-Runge-Lenz vector. It shall be shown that this vector is conserved for Coulomb potential (its commutator with Hamiltonian vanishes) and together with angular momentum (operators) forms so(4) or so(3,1) algebra (after restricting on a subspace of energy eigenstates with either negative or positive energies). In practice is the Laplace-Runge-Lenz vector expressed in terms of c/a operators a this study requires computing a *lot* of commutators, which is to be simplified by expressing the Laplace-Runge-Lenz vector in terms of simpler (ζ) operators.

Results: We have derived the energy spectrum that reproduces the ordinary one in the commutative limit, but has several important features in NC theory. It is regular and no infinite energies (for scattering) are allowed. Instead the spectrum is symmetric around a certain high energy value and there is a surprising set of bound states for *repulsive* potential at ultra high energies. The results confirmed exactly these of differential approach.

Magnetic monopoles

Goal: It was suspected that a certain generalization of considered states should introduce magnetic monopole states to the theory. This suspicion is to be investigated.

Method: When forming physical states an equal number of auxiliary bosonic creation and annihilation operators is being used. This restriction follows from the fact that each of NC coordinates contains exactly one of each and it is reasonable to include only states that become functions of coordinates in the commutative limit. This restriction simplified many calculations, since from it follows that the (omnipresent) radial coordinate operator commutes with all Hilbert space states. To introduce magnetic monopoles states with unequal number of creation as annihilation operators should be considered. To investigate and confirm their presence all calculations of algebraic and dynamical structure have to be revisited and checked for corrections that follow from unequal left and right multiplication with the radial coordinate operator. The obtained results should be compared to those of theory of magnetic monopoles, for example [6].

Results: We have found that the aforementioned generalization really leads to inclusion of magnetic monopoles. This has been proved mostly by relation between the square of velocity and the hamiltonian (where appears an effective monopole potential from static angular momentum), nonzero commutator of the velocity components (proportional to Coulombic field strength) and Casimir operators for the hydrogen atom (in accordance with the ordinary results in the commutative limit). Also the Dirac quantisation condition has been reproduced in a completely new way.

Microscopic black holes

Goal: Analyze consequences of replacing ordinary singularity with a blurred (NC) one on the behavior of (microscopic) black holes.

Method: Follow a method defined in [7]. Matter (energy) density shall be replaced with a NC one, which is defined using coherent states (of Perolomov). Such matter density is then completed into a full stress-energy tensor, which is used to write down the Einstein field equations. They are solved using a 'Schwarzchild-like' ansatz. Using the solution of two things shall be investigated: number of event horizons and the (Hawking) temperature of the black hole radiation.

Results: We have found that depending on the black hole mass there are either two, one or zero event horizons. For mass significantly greater than the Planck mass there is one horizon near the origin and one near to the ordinary one. As the mass becomes smaller these two move toward each other and meet for mass approximately equal to Planck mass. At this stage the temperature drops to zero (instead to grow indefinitely), a microscopic black hole becomes frozen. We have also shown that after adding little mass to it reignites very rapidly. We suggest that microscopic black holes are a very good dark matter candidate (they are minuscule, yet very heavy (compared to the known fundamental particles).

Casimir effect

Goal: Derive Casimir energy for a 1 dimensional system (line) with energy spectrum from NC QM (which has an upper boundary).

Method: The starting point is the velocity operator spectrum, which is inserted into an ordinary theory. To calculate Casimir energy it is summed over all possible frequencies (for a system between two points on a line), contrary to the ordinary case no additional regularization shall be needed.

Results: We have found the Casimir energy of a line divided into three parts by two points. Both cases with outside field present and absent have been considered. In the latter case a confinement like terms dominate in long range region. In both cases there are short range terms that are similar to correction terms for (linear) quark potential correction terms of Lüscher terms [8].

Summary

We have investigated a model of quantum mechanics in noncommutative space. Both kinematic and dynamical problems have been analyzed. We have revealed that certain generalization leads to inclusion of magnetic monopoles into the theory (this approach can be replicated in commutative theory as well). Possible physical consequences of theory have been found in semiclassical approach to (microscopic) black holes and Casimir effect in 1 dimension.

Probably the most important finding is that noncommutativity of the underlying space does not spoil the construction of the theory that much, and if so, only in places where it is expected (and to mentioned welcomed), for example in energy spectra of different systems. Similarly to the construction of ordinary QM, a different mathematical formalism has to be developed, but as far as the physics is concerned, everything behaves very similarly (but without UV ill behavior). We can conclude that noncommutativity of the space is a valid assumption and believe that we have laid a solid ground for a construction of NC QFT.

References

1. **Noncommutative Geometry, A. Connes, Academic Press, London, 1994.**
2. **The construction of noncommutative manifolds using coherent states, H. Grosse, P. Prešnajder, Lett. Math. Phys. 28, 1993.**
3. **Coulomb problem in noncommutative quantum mechanics, V. Gáliková, P. Prešnajder, JMP, 2013.**
4. **Generalized coherent states and their applications, A. Perolomov, Springer, 1986.**
5. **Coherent state induced star product on R^3_λ and the fuzzy sphere, A. B. Hammou, M. Lagraa, M. M. Sheikh-Jabbari, Phys. Rev. D (2002)**

6. Exactly Soluable Nonrelativistic Model of Particles with Both Electric and Magnetic Charges, D. Zwanziger, Physical review 176, 1968.

7. NONCOMMUTATIVE BLACK HOLES, THE FINAL APPEAL TO QUANTUM GRAVITY: A REVIEW, P. Nicolini, IJMP A, 2009.

8. Quark confinement and the bosonic string, M. Lüscher, P. Weisz, JHEP, 2002.

9. Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik, W. Pauli, Zeitschrift für Physik 36, 1926.

10. Spacetime and Geometry: An Introduction to General Relativity, S. Carroll, Addison-Wesley, 2003.

List of publications

1. **T*he velocity operator in quantum mechanics in noncommutative space*, S. Kováčik, P. Prešnajder, published in J.Math.Phys. 54 (2013) 102103.**

   Cited in:

   Spectrum of the three-dimensional fuzzy well, Chandra, N. - Groenewald, H. W. - Kriel, J. N. - Scholtz, F. G. - Vaidya, S.: Published in Journal of Physics A: Mathematical and Theoretical, Vol. 47, No. 44, 2014, Art. No. 445203 - SCI ; SCOPUS

   Eigenvalue problem for radial potentials in space with SU(2) fuzziness, Mirahmadi, M. S. - Fatollahi, A. H.:. Published in Journal of Mathematical Physics, Vol. 55, No. 8, 2014, Art. No. 1.4892999 - SCI ; SCOPUS

2. ***Laplace-Runge-Lenz vector in quantum mechanics in noncommutative space*, V. Gáliková, S. Kováčik, P. Prešnajder, published in J.Math.Phys. 54 (2013) 122106.**

   Cited in:

   Quantum gauge theories on noncommutative three-dimensional space, A. Géré, P. Vitale, J.Ch. Wallet, Published in Phys.Rev. D90 (2014) no.4, 045019.

   On Laplace-Runge-Lenz Vector as Symmetry Breaking order parameter in Kepler Orbit and Goldstone Boson, M. Amri, e-Print: arXiv: 1405.2052 [physics.gen-ph].

   MICZ Kepler Systems in Noncommutative Space and Duality of Force Laws, P. Guha, E. Harikamur, N.S. Zuhair, Published in Int.J.Mod.Phys. A29 (2014) no.32, 1450187.

   Dynamical Symmetry of the Zwanziger problem in Non-commutative Quantum Mechanics, J. Rajhans, e-Print: arXiv: 1412.1149 [math-ph].

   Noncommutative $\mathbb{R}$*^d via closed star product*,V.G. Kupriyanov, P. Vitale, e-Print: arXiv: 1502.06544 [hep-th].

   Fradkin-Bacry-Ruegg-Souriau vector in kappa-deformed space-time, P. Guha, E. Harikamur, N.S.

Zuhair, Published in Eur.Phys.J.Plus 130 (2015) 205.

Physics of the so_q (4) hydrogen atom, P.G. Castro, R. Kullock, Published in Theor.Math.Phys. 185 (2015) no.2, 1678-1684.

Noncommutative R-d via closed star product, Kupriyanov, V. G. - Vitale, Published in Journal of High Energy Physics, No. 8, (2015(, Art. No. 024 - SCI ; SCOPUS

3. ***Laplace-Runge-Lenz Vector in Quantum Mechanics in Noncommutative Space*, V. Gáliková, S. Kováčik, P. Prešnajder, published in Acta Polytech. 54 (2014) no.2, 149–155.**
4. ***Noncommutativity inspired Black Holes as Dark Matter Candidate*, S. Kováčik, e-Print: arXiv: 1504.04460 [gr-qc].**
5. ***Quantum Mechanics in Noncommutative space*, V. Gáliková, S. Kováčik, P. Prešnajder, published in Acta Phys.Slov. 65 (2015) no.3, 153 – 234.**
6. ***NC plane waves, Casimir effect and flux tube potential with Lüscher terms*, S. Kováčik, P. Prešnajder, e-Print: arXiv:1603.00292 [quant-ph].**
7. ***Magnetic monopoles in noncommutative quantum mechanics*, S. Kováčik, P. Prešnajder, e-Print: arXiv:1604.05968 [physics.gen-ph].**
8. ***Princíp neurčitosti v kvantovej mechanike v priestore s nekomutatívnou geometriou*, S. Kováčik, Published report at (Slovak) scientific conference, 2012.**
9. ***Alternatívny opis magnetických monopólov v kvantovej mechanike v komutatívnom a nekomutatívnom priestore*, S. Kováčik, Published report at (Slovak) scientific conference, 2013.**
10. ***Algebraické odvodenie spektra atómu vodíka v nekomutatívnej QM*, S. Kováčik, Published report at (Slovak) scientific conference, 2014.**