Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

Mgr. Michal Kotrbčík

Autoreferát dizertačnej práce

Maximum genus and its generalizations

na získanie vedecko-akademickej hodnosti philosophiæ doctor
v odbore doktorandského štúdia: 9.2.1. informatika

Bratislava 2013

Dizertačná práca bola vypracovaná v internej forme doktorandského štúdia na Katedre informatiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave.

Predkladateľ: Mgr. Michal Kotrbčík
Katedra informatiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave
Mlynská dolina
842 48 Bratislava

Školiteľ: Prof. RNDr. Martin Škoviera, PhD.
Katedra informatiky FMFI UK
Bratislava

Oponenti: ....

....

....

....

Obhajoba dizertačnej práce sa koná dňa .... o ......
pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie dňa ....
v študijnom odbore 9.2.1. Informatika

na ....

Predseda odborovej komisie:
Prof. RNDr. Branislav Rovan, PhD.
Fakulta matematiky, fyziky a informatiky
Univerzity Komenského
Mlynská dolina
842 48 Bratislava

This thesis deals with the maximum genus problem and a novel topological graph invariant, locally-maximal genus in orientable surfaces. The maximum genus of a graph G is the largest integer g such that the graph G has a cellular embedding in the surface of genus g. Maximum genus, denoted by $\gamma_M(G)$, was formally introduced by Nordhaus, Stewart, and White [16] in 1971, and immediately attracted considerable attention. A graph is called upper embeddable if G has embedding with one or two faces. Among the early results on maximum genus have been the proofs of upper-embeddability of complete [16], complete bipartite [17], and complete multipartite graphs [13], and hypercubes [23]. Most of these results use the Edge-adding technique due to Ringeisen [17], which now became part of the folklore. If an edge is added into an embedding of a graph, then there are two possibilities: either the edge splits some face into two faces and the genus remains unchanged, or the edge merges two faces into one and the genus is raised by one. The Ringeisen's result then asserts that the first case occurs if and only if both ends of the edge are added into corners of a single face. When the ends of the edge are added into corners of two distinct faces, then the second case occurs and the genus is raised by one. These early, but fundamental, results were followed by a combinatorial min-max characterization of the maximum genus due to Khomenko et al. [12] and Xuong [21], respectively Khomenko and Glukhov [11] and Nebeský [15]. These results imply that the maximum genus problem is in $NP \cap coNP$. Moreover, two polynomial-time algorithms for the maximum genus were developed, one by Glukhov [6] and the other by Furst et al. [5]. Since then, a significant effort was invested into proving that various graph classes are upper embeddable (such as cyclically 4-edge connected graphs [22] or loopless graphs with diameter 2 [18]) and into obtaining good lower bounds for the maximum genus in graphs classes which are in general not upper embeddable (such as 2 and 3-edge connected graphs [2, 3]). Our main results on the maximum genus are the following:

1) For any graph G we have $\beta(G) - 2\gamma_M(G) \leq \mu(G) \leq \beta(G) - \gamma_M(G)$, where $\mu(G)$ denotes the cycle-packing number of G, the maximum number of vertex-disjoint cycles in G, and both bounds are tight for infinitely-many graphs. Note that $\mu(G)$ is an important NP-complete graph invariant. The result was proved in [1].

2) Let $\nu(G)$ denote the size of a maximum matching in G and $J(G, \mathcal{B})$ the intersection graph of a cycle base $\mathcal{B}$ of G where two elements of the base are adjacent if and only if they have a vertex in common. Note that the elements of $\mathcal{B}$ are graphs with nonzero even valency. With this notation we have the following result [2]. Let G be a connected graph. Then G has a cycle basis $\mathcal{B}$ with $\nu(J(G, \mathcal{B})) = k$ if and only if $\gamma_M(G) \leq k \leq \lfloor \beta(G)/2 \rfloor$.

3) We devise two novel very simple greedy polynomial-time 2-approximation algorithms. The algorithms can be readily implemented and are significantly simpler that both the existing precise algorithms [6, 5] and the existing 4-approximation algorithm [1]. One of these algorithms shows that the classical ideas dating back to

[12] and [21] can be used to efficiently approximate the maximum genus.

The second part of the thesis deals with locally-maximal embeddings, a new concept introduced by the author. An embedding in an orientable surface is called locally-maximal if its genus cannot be raised by any elementary operation. Locally-maximal genus of a graph G is the minimum genus of a locally-maximal embedding of G. We show that this concept is independent from the choice of the elementary operation, as long as we restrict to moving a single edge end in a rotation, moving both edge ends in their respective rotations, or interchanging the position of ends of two edges in some rotation. Moreover, the locally-maximal embeddings are exactly the embeddings that can be output by the greedy algorithm for maximum genus that repeatedly try to raise the genus by employing an elementary operation. Locally-maximal embeddings are related to existing proofs of Duke's interpolation theorem (see [4]), which is usually proved by considering a suitable elementary operation that does not change the genus by more than one, see the proofs in [4, 19, 9, 14] and [20]. Moreover, the set of all embeddings of a graph carries inherent structure given by adjacency of embeddings, where two embeddings are adjacent if one can be obtained from the other by a single application of an elementary operation. Graphs defined by such adjacencies, *stratified graphs*, have been investigated before [8, 7, 10] and locally-maximal embeddings correspond to endpoints of paths always moving to a higher level of the graph.

We investigate locally-maximal embeddings, their properties, and their relation with minimum and maximum genus. Among the obtained results are a lower bound on the locally-maximal genus in terms of the cycle-packing number, several constructions of locally-maximal embeddings, characterizations of all planar and toroidal locally-maximal embeddings, and determination of the locally-maximal genus of complete and complete equipartite graphs and hypercubes. Furthermore, we show that the mentioned greedy algorithm raising the genus whenever possible is in fact a 2-approximation algorithm for the maximum genus (it is the second algorithm mentioned in 3) above). Among the most prominent results is the inequality

$$\gamma(G) \leq \beta'(G)/2 \leq \gamma_L(G) \leq \gamma_M(G),$$

which holds for every connected graph and where $\gamma(G)$ is the minimum genus of G, $\beta'(G)$ is defined by $\beta'(G) = \beta(G) - \mu(G)$, and $\gamma_L(G)$ is the locally-maximal genus of G.

Locally-maximal embeddings have a rich structure and interesting links with other parameters both inside and outside of the topological graph theory. Therefore, we expect their study will prove fruitful and will reveal new information about the structure of the set of all embeddings of a graph. To this end we provide several open problems and directions for the future research.

Bibliography

[1] J. Chen. Algorithmic graph embeddings. *Theoretical Comp. Sci.*, 181:247–266, 1997.

[2] J. Chen, D. Archdeacon, and J. L. Gross. Maximum genus and connectivity. *Discrete Math.*, 149:19–29, 1996.

[3] J. Chen, S. P. Kanchi, and J. L. Gross. A tight lower bound on the maximum genus of a simplicial graph. *Discrete Math.*, 156:83–102, 1996.

[4] R. A. Duke. The genus, regional number, and the Betti number of a graph. *Canad. J. Math.*, 18:817–822, 1966.

[5] M. L. Furst, J. L. Gross, and L. A. McGeoch. Finding a maximum-genus embedding. *J. Assoc. Comput. Mach.*, 35:523–534, 1988.

[6] A. D. Glukhov. A contribution to the theory of maximum genus of a graph. In *Structure and Topological Properties of Graphs*, pages 15–29. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1981. in Russian.

[7] J. L. Gross and R. G. Rieper. Local extrema in genus-stratified graphs. *J. Graph Theory*, 15:159–171, 1991.

[8] J. L. Gross and T. W. Tucker. Local maxima in graded graphs of embeddings. *Annals of the New York Academy of Sciences*, 319:254–257, 1979.

[9] J. L. Gross and T. W. Tucker. *Topological Graph Theory*. Wiley, 1987.

[10] J. L. Gross and T. W. Tucker. Stratified graphs for imbedding systems. *Discrete Math.*, 143(1-3):71–85, 1995.

[11] N. P. Khomenko and A. D. Glukhov. Single-component 2-cell embeddings and maximum genus of a graph. In *Some topological and combinatorial properties of graphs*, pages 5–23. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1980. in Russian.

[12] N. P. Khomenko, N. A. Ostroverkhy, and V. A. Kuzmenko. The maximum genus of a graph. In *ϕ-Transformations of Graphs*, pages 180–207. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1973. in Ukrainian with English summary.

[13] H. V. Kronk, R. D. Ringeisen, and A. T. White. On 2-cell imbeddings of complete n-partite graphs. *Colloq. Math.*, 36(2):295 – 304, 1976.

[14] B. Mohar and C. Thomassen. *Graphs on Surfaces*. The Johns Hopkins University Press, 2001.

[15] L. Nebeský. A new characterization of the maximum genus of a graph. *Czechoslovak Math. J.*, 31(106):604–613, 1981.

[16] E. A. Nordhaus, B. M. Stewart, and A. T. White. On the maximum genus of a graph. *J. of Combinatorial Theory se. B*, 11(3):258 – 267, 1971.

[17] R. D. Ringeisen. Determining all compact orientable 2-manifolds upon which $k_{m,n}$ has 2-cell imbeddings. *J. Combinatorial Theory ser. B*, 12(2):101 – 104, 1972.

[18] M. Škoviera. The maximum genus of graphs of diameter two. *Discrete Math.*, 52:124–146, 1991.

[19] S. Stahl. Generalized embedding schemes. *J. Graph Theory*, 2:41–52, 1978.

[20] A. T. White. *Graphs of Groups on Surfaces: Interactions and Models.* North Holland, 2001.

[21] N. H. Xuong. How to determine the maximum genus of a graph. *J. Combinatorial Theory ser. B*, 26:217–225, 1979.

[22] N. H. Xuong. Upper-embeddable graphs and related topics. *J. Combinatorial Theory ser. B*, 26:226–232, 1979.

[23] J. Zaks. The maximum genus of cartesian products of graphs. *Canad. J. Math.*, 26:1025–1035, 1974.

List of published papers of the author related with the thesis

[1] M. Kotrbčík. A note on disjoint cycles. *Inform. Process. Lett.*, pages 135–137, 2012.

[2] M. Kotrbčík and M. Škoviera. Matchings, cycle bases, and the maximum genus of a graph. *Electronic J. Combin.*, 19, 2012. ♯P3.

[3] M. Kotrbčík. Maximum genus of regular graphs. *Electron. Notes in Discrete Math.*, 38:535 – 540, 2011.