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Mgr. Ján Komadel

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Predkladateľ: Mgr. Ján Komadel
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave
Mlynská dolina, 842 48 Bratislava

Školiteľ: doc. Ing. Aleš Černý, PhD.
Faculty of Finance
Cass Business School
106 Bunhill Row, London
EC1Y 8TZ, United Kingdom

Študijný odbor: 1114 Aplikovaná matematika

Predseda odborovej komisie:

prof. RNDr. Daniel Ševčovič, DrSc.
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave
Mlynská dolina, 842 48 Bratislava

1 Introduction

In finance, it is often one's aim to find an optimal strategy of execution. Common objectives include maximizing revenue, profit, or utility in a broader sense, and minimizing risk. Naturally, mathematical optimization is a key instrument in finding the optimal strategy. We study two applications of optimization in finance. The first one is the optimal liquidation problem where the investor aims to maximize the expected revenue from selling a certain amount of an asset while the received price at each time is adversely affected by the amount sold at this time. The studied optimal liquidation problem and our results are described in Section 2. The other application is option hedging in an incomplete market where the mean squared hedging error of a discrete delta hedging strategy is analyzed for Asian options. We present the studied problem and our results in Section 3.

2 Optimal liquidation

The topic of optimal liquidation or optimal trade execution addresses the question of how to sell a given amount of an asset, maximizing the investor's utility from the sale, when the execution price is adversely affected by the sale. Alternatively, the theory can also be applied to the optimal purchase of an asset rather than its sale. The seminal papers in this field are by Bertsimas and Lo [3] and Almgren and Chriss [1]. In these works the asset price is assumed to be a martingale and the pressure to liquidate is exogenous. In particular, the authors use a fixed date by which the whole position must be liquidated.

We study the problem of optimal liquidation as it was first formulated by Černý [6] and this formulation differs from most of optimal liquidation literature in two related aspects. Firstly, the pressure to liquidate is given endogenously and, secondly, the liquidation horizon is stochastic and it is determined as a part of the optimal strategy. The model allows different kinds of the endogenous pressure to liquidate including the asset price falling on average or time discounting.

2.1 Formulation of the problem

We consider a problem of maximizing the expected revenue from liquidating a position $Z_0 = z > 0$ of an asset whose price is adversely affected by the amount being sold. The so called *unaffected price* process, i.e. the price prevailing in the market in absence of our trading, is given by the geometric Brownian motion

$$dS_t = \lambda S_t dt + \sigma S_t dW_t, \tag{1}$$

with an initial price $S_0 = s > 0$, and the amount of the asset yet to be sold Z_t has the dynamics

$$dZ_t = \big(rZ_t - v_t\big)dt, \tag{2}$$

where r is the growth rate of the asset and v_t is the selling rate. The objective function, which is to be maximized over selling strategies v, is

$$E\left(\int_0^{T(Z=0)} e^{-\rho t} v_t\big(S_t - \eta v_t\big)dt\right),$$

where ρ is the discount rate and $T(Z = 0)$ denotes the first time t such that $Z_t = 0$, i.e. the time when the whole amount is sold. The use of the stopping time $T(Z = 0)$ is a novel feature in optimal liquidation where the time horizon is typically given exogenously. Our approach rules out short sales once the inventory is disposed of but it leaves open the possibility of intermediate purchases. However, these turn out to be never optimal.

Note that the price received by the investor is negatively affected by the amount being sold. This is represented by the term ηv_t by which the unaffected price S_t is reduced. This represents the instantaneous effect, or *temporary impact*, of selling on the price and it creates an incentive to sell at a low rate as opposed to the pressure to liquidate.

As shown in [6], the Hamilton-Jacobi-Bellman (HJB) equation for the value function

$$V(s, z) = \sup_v E\left(\int_0^{T(Z=0)} e^{-\rho t} v_t\big(S_t - \eta v_t\big)dt\right) \tag{3}$$

subject to the dynamics (1), (2), leads to the partial differential equation

$$\frac{1}{2}s^2\sigma^2 V_{ss} + \lambda s V_s + rzV_z - \rho V + \frac{(s - V_z)^2}{4\eta} = 0 \tag{4}$$

for $s > 0$, $z > 0$, with the condition $V(s, 0) = 0$ corresponding to the revenue being zero whenever the amount of the asset is $z = 0$, regardless of the price.

Employing the scaling

$$V(s, z) = \frac{s^2}{\eta\sigma^2}u(x), \quad x = \eta\sigma^2\frac{z}{s} \tag{5}$$

the PDE (4) can be reduced to the ordinary differential equation for $x > 0$

$$x^2 u'' = axu' + bu - \frac{1}{2}(u' - 1)^2, \tag{6}$$

$$u(0) = 0, \tag{7}$$

where we define

$$a = \frac{2}{\sigma^2}\left(\lambda - r + \sigma^2\right), \qquad b = -\frac{2}{\sigma^2}\left(2\lambda - \rho + \sigma^2\right). \tag{8}$$

We make the assumption throughout that $a + b > 0$ which translates to

$$\rho > \lambda + r, \tag{9}$$

creating the endogenous pressure to liquidate. Were this condition not satisfied, the investor could profit from postponing the sale infinitely.

We denote the initial value problem (6), (7) by IVP_0

$$\begin{aligned} x^2 u'' &= axu' + bu - \frac{1}{2}(u' - 1)^2, \\ u(0) &= 0, \end{aligned} \tag{IVP$_0$}$$

This problem is studied in Brunovský, Černý and Winkler [4] and in Quittner [12] where its severe singularity is shown. For $a + b > 0$ IVP_0 has infinitely many solutions with identical asymptotics near 0 given by the formal power series

$$h_n(x) = \sum_{i=0}^{n} k_i x^{1+i/2},\ n \in \mathbb{N}, \tag{10}$$

with $k_0 = 1$, $k_1 = -\frac{2}{3}\sqrt{2(a+b)}$ and the other k_i obtained from a recursive relationship.

Brunovský, Černý and Winkler [4, Proposition 5.1] show that IVP_0 does, however, have a unique solution under additional conditions and they also prove certain characteristics of the solution which we summarize in the following proposition.

Proposition 2.1. *Under the assumption $a + b > 0$ there is a unique solution of IVP_0 denoted by u_∞ satisfying $u_\infty \in \mathcal{C}^0[0,\infty) \times \mathcal{C}^2(0,\infty)$ and*

$$0 \le u_\infty(x) \le x \textit{ for } x > 0.$$

The solution u_∞ further satisfies $u'_\infty(0) = 1$, $u'_\infty(x) > 0, u''_\infty(x) < 0$, $u'''_\infty(x) > 0$ for all $x > 0$ as well as $u'_\infty(x) \searrow 0$ for $x \to \infty$.

At this time, two questions are still open. The first being that the HJB equation is in general only a necessary condition and thus it needs to be verified whether the found solution $V(s, z)$ is indeed the value function. The second question is how can the unique solution u_∞ from Proposition 2.1 be calculated. We address these questions in the following subsections.

2.2 Optimality

Inspired by Proposition 2.1 we add to IVP_0 a boundary condition for $x \to \infty$ and define

$$\begin{aligned} x^2 u'' &= axu' + bu - \frac{1}{2}(u'-1)^2, \\ u(0) &= 0, \qquad u'(\infty) = 0, \end{aligned} \tag{$\mathrm{BVP}_{[0,\infty)}$}$$

where we write $u'(\infty) = \lim_{x\to\infty} u'(x)$ whenever the limit on the right-hand side exists.

In [5, Proposition 6.1] we show that the addition of the boundary condition has the effect of uniquely determining the solution of IVP_0 established in Proposition 2.1, i.e. u_∞ from Proposition 2.1 is the unique solution $\mathrm{BVP}_{[0,\infty)}$.

The next theorem confirms that by solving $\mathrm{BVP}_{[0,\infty)}$ and subsequent use of scaling (5) we indeed find the value function of the optimal liquidation problem. Furthermore, the theorem characterizes the optimal strategy v^* which turns out to be nonnegative. This means that it is never optimal to acquire more of the asset even though our setting only rules out short sales and intermediate purchases are admissible.

Theorem 2.2. *Assume* (9). *Let u_∞ be the unique solution of $BVP_{[0,\infty)}$, with a, b given by* (8). *Then the function $V(s,z) = \frac{s^2}{\eta\sigma^2} u_\infty\left(\eta\sigma^2 \frac{z}{s}\right) \le sz$ is the value function of the optimization* (3) *and*

$$v_t^* = \frac{1}{2\eta}\Big(S_t - V_z\left(S_t, Z_t^*\right)\Big) = \frac{S_t}{2\eta}\left(1 - u'_\infty\left(\eta\sigma^2 \frac{Z_t^*}{S_t}\right)\right) \ge 0 \tag{11}$$

is the optimal control among all admissible controls.

According to Theorem 2.2 the implementation shortfall $sz - V(s,z)$ is nonnegative in our setting which rules out short sales. This agrees with intuition and differs from results of [13] where the investor can benefit from short sales and subsequent purchases in a bearish market.

From (10) the asymptotic relative implementation shortfall for small values of the inventory z is

$$I(s,z) = \frac{sz - V(s,z)}{sz} = \frac{s - \frac{V(s,z)}{z}}{s} = \frac{4}{3}\sqrt{\eta(\rho - \lambda - r)\frac{z}{s}} + \mathcal{O}(z). \tag{12}$$

It can also be interpreted as the relative difference between the initial price s and the average realized price $V(s,z)/z$. For this reason $I(s,z)$ is often referred to in empirical literature as the *price impact*. The result (12) agrees with the square root law, observed in empirical studies such as [15], which says that the price impact is proportional to the square root of the total trade size z.

2.3 Computation of the solution

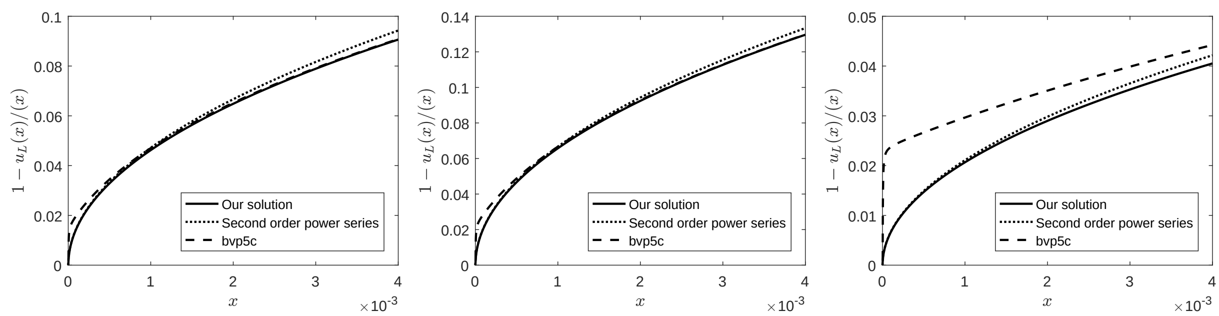
The first step towards computing the solution u_∞ of BVP$_{[0,\infty)}$ is to truncate x to a finite interval $[0, L]$, $L < \infty$, defining the boundary value problem BVP$_{[0,L]}$

$$\begin{aligned} &x^2 u'' = axu' + bu - \frac{1}{2}(u' - 1)^2, \\ &u(0) = 0, \qquad u'(L) = 0. \end{aligned} \tag{BVP$_{[0,L]}$}$$

The next theorem shows that BVP$_{[0,L]}$ has a unique solution $u_L(x)$ which tends pointwisely to $u_\infty(x)$, the solution of BVP$_{[0,\infty)}$, as $L \to \infty$. This justifies looking for $u_\infty(x)$ by solving the truncated problem BVP$_{[0,L]}$ with L sufficiently large.

Theorem 2.3. *Let $a + b > 0$. For given $L > 0$, BVP$_{[0,L]}$ has a unique solution $u_L \in C^2\big((0, L]\big) \cap C^0\big([0, L]\big)$ such that $0 \le u_L(x) \le x$ for all $x \in [0, L]$. The solution u_L is strictly increasing, concave and satisfies $u_{L_1}(x) \le u_{L_2}(x)$ for $L_1 \le L_2,\ 0 \le x \le L_1$. Furthermore, $\lim_{L\to\infty} u_L(x) = u_\infty(x)$ for $0 \le x < \infty$, where u_∞ is the unique solution of BVP$_{[0,\infty)}$.*

Literature studies numerical methods for boundary value problems of similar type as BVP$_{[0,L]}$ (cf. [11, 17]), however, the singularity caused by the nonlinear of the ODE in BVP$_{[0,L]}$ is more severe than singularities typically considered and thus standard methods for numerical treatment of BVPs fail in this case. This is illustrated in Figure 1 where we observe that the Matlab routine bvp5c is unable to capture the dynamics of the solution of BVP$_{[0,L]}$ near the singularity at $x = 0$. The displayed quantity $1 - u_L(x)/(x)$ represents approximate implementation shortfall $I(s, z)$. The actual solution $u_L(x)$ is approximated by the second-order asymptotic power series $h_2(x) = x - \frac{2}{3}\sqrt{2(a+b)}x^{3/2}$ defined in (10).



(a) Parametrization 1. **(b)** Parametrization 2. **(c)** Parametrization 3.

Figure 1: Comparison of BVP$_{[0,L]}$ solution to solution from Matlab routine bvp5c near the singularity at $x = 0$. The displayed quantity $1 - u_L(x)/(x)$ represents approximate implementation shortfall.

To overcome the problem with the severe singularity it turns out to be advantageous to introduce a time dimension into the optimal liquidation problem (3) which corresponds to setting a finite time horizon. In this way one obtains instead of the ODE (6) a parabolic partial differential equation

$$w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2 \tag{13}$$

for the function $w(t,x)$. For the PDE (13) we define on $[0,\infty) \times [0,L]$ the boundary value problem $\underline{\text{BVP}}^t_{[0,L]}$ by

$$\begin{gathered} w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2, \\ w(t,0) = 0, \qquad w_x(t,L) = 0, \qquad w(0,x) = 0. \end{gathered} \tag{$\underline{\text{BVP}}^t_{[0,L]}$}$$

In addition, we define a second problem, $\overline{\text{BVP}}^t_{[0,L]}$, by replacing the zero initial condition with $w(0,x) = x$

$$\begin{gathered} w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2, \\ w(t,0) = 0, \qquad w_x(t,L) = 0, \qquad w(0,x) = x. \end{gathered} \tag{$\overline{\text{BVP}}^t_{[0,L]}$}$$

By $\text{BVP}^t_{[0,L]}$ we refer to either of the problems $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$.

In the following theorem we show that as $t \to \infty$ the solutions of $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$ tend to the solution of the time homogenous problem $\text{BVP}_{[0,L]}$ monotonically from below, resp. from above, see Figure 2 in the following chapter.

Theorem 2.4. *For given L the problems $\underline{BVP}^t_{[0,L]}$ and $\overline{BVP}^t_{[0,L]}$ have a unique solution in $\mathcal{C}^{1,2}\big((0,\infty) \times (0,L]\big) \cap \mathcal{C}\big([0,\infty) \times [0,L]\big)$. These solutions, denoted by $\underline{w}$ and $\overline{w}$ respectively, satisfy*

$$\begin{array}{ccccc} 0 \le \underline{w}(t,x) & \le & u_L(x) & \le & \overline{w}(t,x) \le x \\ \dfrac{\partial \overline{w}(t,x)}{\partial t} & \le & 0 & \le & \dfrac{\partial \underline{w}(t,x)}{\partial t} \end{array}$$

and $\lim_{t\to\infty} \overline{w}(t,x) = \lim_{t\to\infty} \underline{w}(t,x) = u_L(x)$.

Table 1 schematically shows the procedure which connects the numerically amenable finite horizon problems to value function (3) and which will be used in numerical computations.

Table 1: Computation of the value function from the finite horizon problems.

$\underline{\text{BVP}}^t_{[0,L]}$ or $\overline{\text{BVP}}^t_{[0,L]}$	$\xrightarrow[t\to\infty]{}$	$\text{BVP}_{[0,L]}$	$\xrightarrow[L\to\infty]{}$	$\text{BVP}_{[0,\infty)}$	$\xrightarrow[(5)]{}$	(3)
$\underline{w}(t,x)$ or $\overline{w}(t,x)$	$\xrightarrow[t\to\infty]{}$	$u_L(x)$	$\xrightarrow[L\to\infty]{}$	$u_\infty(x)$	$\xrightarrow[(5)]{}$	$V(s,z)$

2.4 Numerical results

2.4.1 Solving $\mathbf{BVP}^t_{[0,L]}$

The finite horizon problems $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$ are given by the PDE (13)

$$w_t = x^2 w_{xx} - axw_x - bw + \frac{1}{2}(w_x - 1)^2$$

which we treat numerically. For the spatial variable $x \in [0, L]$ we employ a non-equidistant partition $0 = x_0 < x_1 < \cdots < x_N = L$, where the partition points are defined as

$$x_j = e^{\xi_j} - 1 - \xi_j + \xi_j^{3/2}, \qquad j = 0, 1, \ldots, N \tag{14}$$

with $\{\xi_j\}_{j=0}^N$ being equidistant, i.e. $\xi_j = jL^*/N$ with L^* such that $x_N = e^{L^*} - 1 - L^* + (L^*)^{3/2} = L$. The used non-equidistant partition is finer for small values of x and coarser for larger values. For the time variable $t \in [0, T]$ we use an equidistant partition with the time step $h = T/M$. The used explicit Euler scheme for PDE (13) reads

$$\begin{aligned} w_j^{i+1} = w_j^i + h \Bigg[& \frac{2\,x_j^2}{x_{j+1} - x_{j-1}} \left(\frac{w_{j+1}^i - w_j^i}{x_{j+1} - x_j} - \frac{w_j^i - w_{j-1}^i}{x_j - x_{j-1}} \right) \\ & - a\,x_j \frac{w_{j+1}^i - w_{j-1}^i}{x_{j+1} - x_{j-1}} - b\,w_j^i + \frac{1}{2} \left(\frac{w_{j+1}^i - w_{j-1}^i}{x_{j+1} - x_{j-1}} - 1 \right)^2 \Bigg] \end{aligned} \tag{15}$$

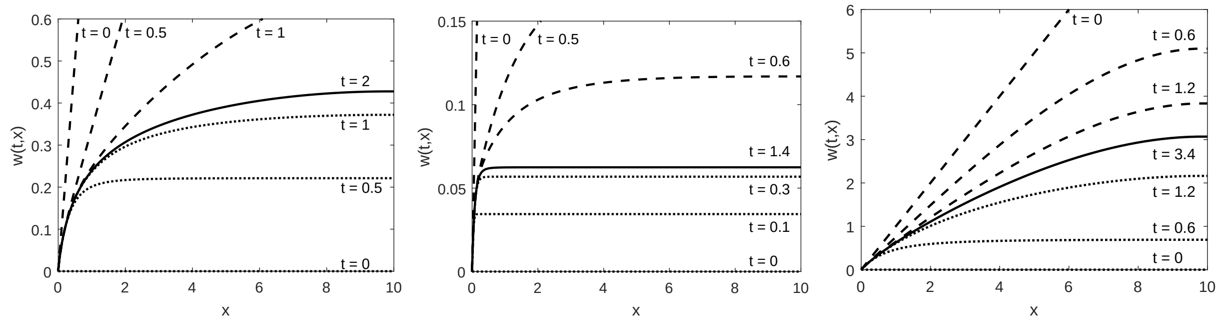
for $i = 0, 1, \ldots, (M-1)$ and $j = 1, 2, \ldots, (N-1)$.

The boundary conditions, given for both $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$ by $w(t, 0) = 0$ and $w_x(t, L) = 0$, dictate $w_0^i = 0$, $w_N^i = w_{N-1}^i$, $i = 0, 1, \ldots, M$, while $\underline{\text{BVP}}^t_{[0,L]}$ has the zero initial condition $w_j^0 = 0$ and $\overline{\text{BVP}}^t_{[0,L]}$ has $w_j^0 = x_j$, $j = 0, 1, \ldots, N$.

Table 2: Parameter values used in numerical examples.

	σ	λ	r	ρ	a	b
Parametrization 1	0.2	0	0	0.05	2	0.5
Parametrization 2	0.2	-0.1	0	0	-3	8
Parametrization 3	0.2	0.03	0.01	0.05	3	-2.5

As claimed by Theorem 2.4 the solutions of $\underline{\text{BVP}}^t_{[0,L]}$ and $\overline{\text{BVP}}^t_{[0,L]}$ should monotonically converge to u_L, the solution of $\text{BVP}_{[0,L]}$, from below and above, respectively, as t increases. This can be observed in Figure 2 for the three sets of parameters listed in Table 2.



(a) Parametrization 1. **(b)** Parametrization 2. **(c)** Parametrization 3.

Figure 2: Solutions of $\underline{\mathrm{BVP}}^t_{[0,L]}$ (dotted) and $\overline{\mathrm{BVP}}^t_{[0,L]}$ (dashed) for $L = 10$ and different values of t. The solid lines represent solution of $\mathrm{BVP}_{[0,L]}$.

2.4.2 Solving $\mathbf{BVP}_{[0,\infty)}$

As shown in Table 1, a sufficiently precise approximation of u_∞ can be obtained by solving the finite horizon problems $\mathrm{BVP}^t_{[0,L]}$ with increasing t and L. We use four nested loops, going from the innermost one moving outwards, to determine the number of time steps M (and thus also the time horizon $T = Mh$), the number of x-partition points N, the length of the spatial interval L, and the time step h.

Our goal is to achieve sufficient precision of $u_\infty(x)$ on the interval $[0, 1]$ and in each of the four loops we look at the relative implementation shortfall (12)

$$I(s,z) = \frac{sz - V(s,z)}{sz} = \frac{sz - \frac{s^2}{\eta\sigma^2} u\left(\eta\sigma^2 \frac{z}{s}\right)}{sz} = 1 - \frac{u(x)}{x}.$$

for two consecutive approximations of the solution. For small x we aim for a low relative difference in the shortfall $I(s, z)$ while for larger x we aim for low absolute difference.

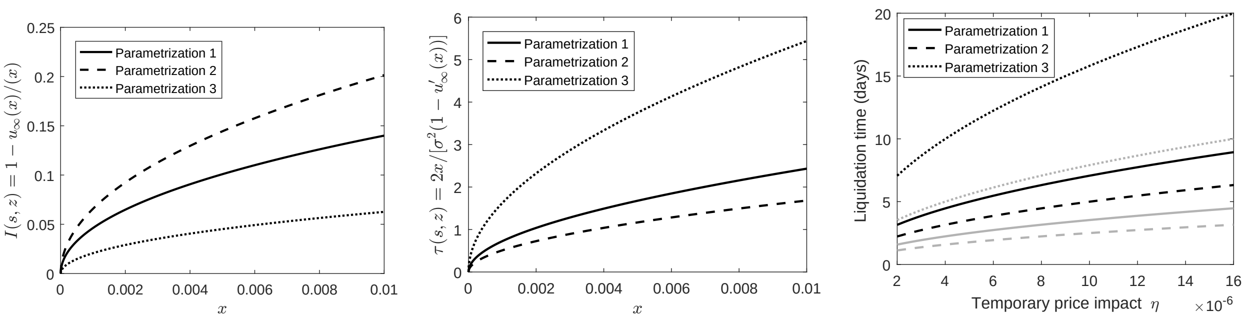
2.4.3 Results for the optimal liquidation problem

Panel (a) of Figure 3 shows the relative implementation shortfall $I(s, z) = 1 - u_\infty(x)/x$ for the three parametrizations in Table 2. Panel (b) of the same figure shows the quantity

$$\tau(s,z) = \frac{z}{v(s,z)} = \frac{2\,x}{\sigma^2\big(1 - u'_\infty(x)\big)}$$

which expresses the time the liquidation of amount z of the asset would take if the the selling rate stayed constant at $v(s, z)$. Observe that both $I(s, z)$ and $\tau(s, z)$ increase with x which is proportional to the trade size z for a fixed asset.

Panel (c) of Figure 3 compares $\tau(z, s)$ to actual average time to liquidation $T(Z = 0)$, based on 10 000 simulations, for changing values of the temporary price impact parameter η. The actual time to liquidation $T(Z = 0)$ is approximately twice as long as $\tau(z, s)$



(a) $I(s, z)$. **(b)** $\tau(s, z)$. **(c)** $T(Z = 0)$ and $\tau(z, s)$.

Figure 3: (a) Relative implementation shortfall $I(s, z)$. (b) time to liquidation assuming constant liquidation speed and no accruing interest $\tau(s, z)$. (c) Actual average time to liquidation, $T(Z = 0)$ (black) and approximate time to liquidation $\tau(z, s)$ (grey) for changing values of the temporary price impact parameter η.

because the actual liquidation speed (11) decreases as the asset is being sold.

2.5 Conclusion

In this part of the thesis we dealt with the problem of optimal liquidation, using the formulation of the problem which first appeared in [6]. The main feature that differentiates this formulation from the literature is that in our case the pressure to liquidate is given endogenously which results in a stochastic liquidation horizon, given as a part of the optimal strategy. Moreover, our formulation rules out short sales and we find that intermediate purchases turn out to be never optimal, even though they are permitted.

The optimal liquidation problem leads to a severely singular initial value problem IVP_0 which has been studied in [4] and [12] and for which standard numerical methods fail. We presented a method of solving IVP_0 which consists of solving related boundary value problems $\text{BVP}^t_{[0,L]}$ and stretching the finite horizon t and the length of the spatial interval L. We demonstrated that this method produces stable numerical solutions which agree with the known asymptotics near zero.

Numerical approximations of the solutions of IVP_0 were then used to examine the relative implementation shortfall resulting from liquidation which was found to be consistent with the square root law known from empirical literature. Furthermore, we examined the stochastic time to liquidation resulting from the optimal execution strategy, comparing an approximate time to liquidation to estimates of the actual time obtained by simulations and and we studied the process of optimal liquidation by simulating the liquidation process.

3 Quadratic hedging

A simple financial market can be described on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\big)$ by the processes B and S giving the prices of basic assets – a risk-free bond and a stock, respectively. It is convenient to take the bond as numéraire, making the bond price 1 at all times, and work only with the stock price $\tilde{S} = S/B$ expressed in terms of the bond rather than with the original price S expressed in monetary terms.

The value of a *self-financing* portfolio consisting of φ_t stocks and ψ_t can be written as

$$V_t = V_0 + \int_0^t \varphi_\tau d\tilde{S}_\tau,$$

where V_0 is the initial investment needed to start the strategy. A contingent claim H is *attainable* if there exists a self-financing strategy with $V_T = H$. A market is said to be *complete*, if any $\mathcal{F}_T$-measurable claim H is attainable.

In reality, however, one rarely encounters a complete market. For this reason we consider an incomplete market where non-attainable contingent claims exist. These can no longer be hedged perfectly and so it is sensible to look for the best possible hedge according to some criterion. One of the most widely used criteria is based on keeping the self-financing condition but relaxing the replication condition $V_T = H$. Thus, a hedging error, also referred to as *tracking error* in literature (cf. [2]), is introduced

$$H - V_T(V_0, \varphi) = H - V_0 - \int_0^T \varphi_\tau d\tilde{S}_\tau.$$

The aim of *quadratic hedging* or *mean-variance hedging* is to look for a strategy (V_0, φ) which minimizes the *mean squared hedging error* (MSHE)

$$\varepsilon^2(V_0, \varphi) = E\left[\left(H - V_0 - \int_0^T \varphi_\tau d\tilde{S}_\tau\right)^2\right]. \tag{16}$$

over all $V_0 \in \mathbb{R}$ and $\tilde{S}$-integrable processes φ.

We study the MSHE ε^2 for a discretely implemented delta hedging strategy in case of an arithmetic Asian option. Literature finds that the delta hedging strategy performs similarly to the locally and globally optimal strategies in case of independent and identically distributed returns.

3.1 Asian options

Asian options are derivatives whose payoff depends on the average of the underlying stock price over a given period of time. Typically, the arithmetic average is used

$$A_t = \frac{1}{T}\int_0^t S_\tau d\tau,$$

where S_τ is the stock price at time τ.

Under the assumption that the stock price S is governed by the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t W_t, \tag{17}$$

the arithmetic average Asian option price $u(t, S_t, A_t)$ can be shown to solve the equation

$$u_t + rSu_S + \frac{1}{T}Su_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0. \tag{18}$$

The terminal condition is given by the option payoff $u(T, S_T, A_T) = h(S_T, A_T)$ and it depends on the particular type of the Asian option. A *fixed strike* call option with strike K has the payoff $h(S_T, A_T) = (A_T - K)^+$, while a *floating strike* call option has the payoff $h(S_T, A_T) = (S_T - A_T)^+$.

3.2 Discrete delta hedging

If continuous trading is feasible, an Asian option can be perfectly replicated by a portfolio holding $\varphi_t = u_S(t, S_t, A_t)$ stocks at any time. This is called *delta hedging* with u_S being the option delta. In reality, this is not plausible mainly because of transaction costs which make continuous trading infinitely expensive.

We consider a delta hedging strategy implemented at discrete times during $[0, T]$. Let the trading times t_k be given by $t_k = k\Delta t$ for $k = 0, \ldots, n$ with $\Delta t = T/n$. At each t_k the investor rebalances the portfolio so that it contains $\varphi_{t_k} = u_S(t_k, S_{t_k}, A_{t_k})$ units of the stock. This amount is then held until the next trading time t_{k+1}. The mean squared hedging error is then given by

$$\varepsilon^2(V_0, \varphi^\Delta) = E\left[\left(\int_0^T \left[u_S(\tau, S_\tau, A_\tau) - u_S\big(\theta(\tau), S_{\theta(\tau)}, A_{\theta(\tau)}\big)\right] d\tilde{S}_\tau\right)^2\right]. \tag{19}$$

The dependence of the MSHE on the length of the the rehedging interval has been studied for different option types in various models. Bertsimas, Kogan and Lo [2] study for European options in a generalized Black-Scholes model with $dS_t = \mu(t, S_t)S_t dt +$

$\sigma(t, S_t)S_t dW_t$. They derive the approximation of (19)

$$\varepsilon^2(V_0, \varphi^\Delta) = \frac{\Delta t}{2} E\left[\int_0^T \left(\sigma^2(\tau, S_\tau)S_\tau^2 u_{SS}(\tau, S_\tau)\right)^2 d\tau\right] + o(\Delta t). \tag{20}$$

The same approximation was derived independently in Zhang [18].

The results of [2] have been extended to stochastic volatility models by [10], exponential Lévy models by [7] and to general Itō processes with jumps by [14]. Gobet and Temam [9] show for a wider class of options that the rate of convergence is between $\sqrt{\Delta t}$ and Δt, depending on smoothness of the payoff. Their research was followed by [8] where hedging errors of delta-gamma strategies are studied.

3.3 Mean squared hedging error approximation

We heuristically show that under the assumption of the risk-neutral dynamics

$$d\tilde{S}_t = \sigma(\tilde{S}_t)d\tilde{W}_t$$

the mean squared hedging error of a discretely applied delta hedging strategy can be approximated by

$$\varepsilon^2 \approx \frac{\Delta t}{2} E\left[\int_0^T \sigma^4(\tilde{S}_t)u_{SS}^2(t, S_t)dt\right], \tag{21}$$

which, for the setting considered by Bertsimas, Kogan and Lo, coincides with their approximation (20).

In case of an Asian option and the stock price dynamics (17), approximation (21) reads

$$\varepsilon^2 \approx \frac{\Delta t}{2} E\left[\int_0^T \sigma^4 S_t^4 u_{SS}^2(t, S_t, A_t)dt\right]. \tag{22}$$

We show that the expectation in (22) can be evaluated by solving the system

$$u_t + rSu_S + \frac{1}{T}Su_A + \frac{1}{2}\sigma^2 S^2 u_{SS} - ru = 0 \tag{23}$$

$$\left(\sigma^2 S^2 u_{SS}\right)^2 + v_t + \mu S v_S + \frac{1}{T}Sv_A + \frac{1}{2}\sigma^2 S^2 v_{SS} = 0 \tag{24}$$

where $u(t, S_t, A_t)$ is the option price and

$$v(t, S_t, A_t) = E_t\left[\int_t^T \sigma^4 S_\tau^4 u_{SS}^2(\tau, S_\tau, A_\tau)d\tau\right], \tag{25}$$

yielding the zero terminal condition $v(T, S_T, A_T) = 0$.

The dimension of the problem can be reduced to one state variable χ instead of S, A

and the MSHE approximation (22) can evaluated by solving the reduced system

$$f_t(t,\chi) + \frac{1}{2} c\,(t,\chi)\, f_{\chi\chi}(t,\chi) = 0, \tag{26}$$

$$4\Big(f_t(t,\chi)\Big)^2 + g_t(t,\chi) + b\,(t,\chi)\, g_\chi(t,\chi) + \frac{1}{2} c\,(t,\chi)\, g_{\chi\chi}(t,\chi) = 0. \tag{27}$$

instead of the original system (23), (24). The terms $b\,(t,\chi)$ and $c\,(t,\chi)$ are given by

$$b\,(t,\chi) = \left(\chi - \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)(r-\hat{\delta}-\mu), \tag{28}$$

$$\sqrt{c\,(t,\chi)} = \left(\chi - \frac{1-e^{-(r-\hat{\delta})(T-t)}}{(r-\hat{\delta})T}\right)\sigma. \tag{29}$$

Equation (26) for $f(t,\chi)$ is the reduced Asian pricing equation proposed by Večeř (2002) [16] and the terminal condition is given by $f(T,\chi) = \chi^+$ or $f(T,\chi) = (1-\chi)^+$ for the fixed and floating strike call, respectively. The terminal condition for (27) is $g(T,\chi) = 0$.

Having solved the system (26), (27) the MSHE approximation (22) can be evaluated by means of function g as

$$\varepsilon^2 \approx \frac{\Delta t}{2} E\left[\int_0^T \left(\sigma^2 S_\tau^2 u_{SS}(\tau, S_\tau, A_\tau)\right)^2 d\tau\right] = \frac{\Delta t}{2} S_0^2 e^{\left(2\mu+\sigma^2\right)T} g(0,\chi_0). \tag{30}$$

3.4 Numerical results

3.4.1 Computing the option price

Equation (26) does not contain function $g(t,\chi)$ so it can be solves independently of equation (27). For the purpose of numerical treatment we limit the considered range for $\chi \in \mathbb{R}$ to $\chi \in [-\chi_{max}, \chi_{max}]$ and we use equidistant partitions for both variables $\chi_j = -\chi_{max} + j \times \delta\chi$, $j = 0,\dots,n_\chi$, with $\delta\chi = 2\chi_{max}/n_\chi$, and $t_i = i \times \delta t$, $i = 0,\dots,n_t$, with $\delta t = T/n_t$.

Denoting f_j^i, c_j^i the approximations of $f(t_i,\chi_j)$ and $c(t_i,\chi_j)$, respectively, the Crank-Nicolson method for PDE (26) reads

$$\begin{aligned} -\frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2} f_{j-1}^i + \left(1 + \frac{c_j^i}{2}\frac{\delta t}{(\delta\chi)^2}\right) f_j^i - \frac{1}{4} c_j^i \frac{\delta t}{(\delta\chi)^2} f_{j+1}^i \\ = \frac{1}{4} c_j^{i+1} \frac{\delta t}{(\delta\chi)^2} f_{j-1}^{i+1} + \left(1 - \frac{c_j^{i+1}}{2}\frac{\delta t}{(\delta\chi)^2}\right) f_j^{i+1} + \frac{1}{4} c_j^{i+1} \frac{\delta t}{(\delta\chi)^2} f_{j+1}^{i+1} \end{aligned} \tag{31}$$

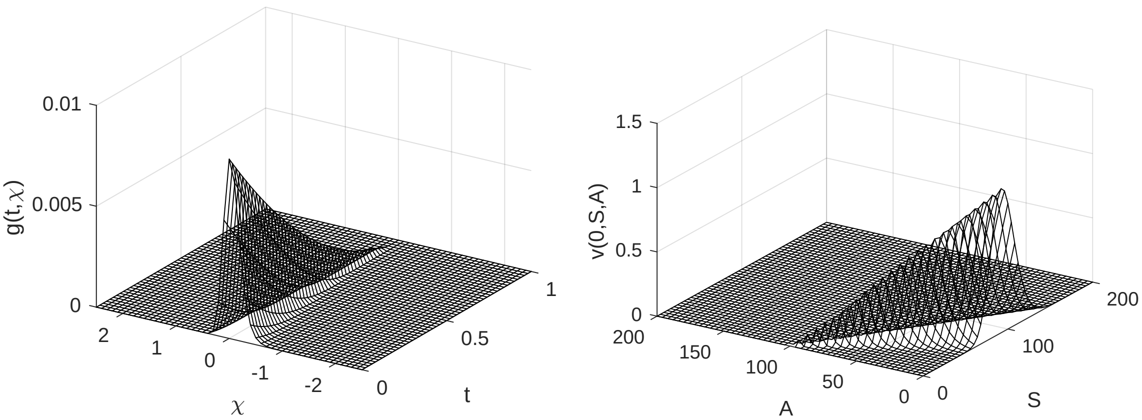
for $i = 0,\dots,(n_t-1)$ and $j = 1,\dots,(n_\chi-1)$, which applies to both fixed and floating strike options. However, the terminal and boundary conditions differ for the two types.

3.4.2 Solving the MSHE equation

Having found a numerical solution $f(t,\chi)$, we can approach to solving equation (27). Denoting g^i_j, b^i_j the approximations of $g(t_i,\chi_j)$ and $b(t_i,\chi_j)$, respectively, the Crank-Nicolson method for PDE (27) reads

$$\begin{aligned} -\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c^i_j}{\delta\chi}-b^i_j\right)g^i_{j-1}+\left(1+\frac{c^i_j}{2}\frac{\delta t}{(\delta\chi)^2}\right)g^i_j-\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c^i_j}{\delta\chi}+b^i_j\right)g^i_{j+1} \\ =\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c^{i+1}_j}{\delta\chi}-b^{i+1}_j\right)g^{i+1}_{j-1}+\left(1-\frac{c^{i+1}_j}{2}\frac{\delta t}{(\delta\chi)^2}\right)g^{i+1}_j \\ +\frac{1}{4}\frac{\delta t}{\delta\chi}\left(\frac{c^{i+1}_j}{\delta\chi}+b^{i+1}_j\right)g^{i+1}_{j+1}+\frac{4}{\delta t}\left(f^{i+1}_j-f^i_j\right)^2. \end{aligned} \tag{32}$$

for $i=0,\dots,(n_t-1)$ and $j=1,\dots,(n_\chi-1)$. The zero terminal condition $g^{n_t}_j=0$, $j=0,\dots,(n_\chi-1)$ is accompanied by zero boundary conditions $g^i_0=0$, $g^i_{n_\chi}=0$, $i=0,\dots,(n_t-1)$.



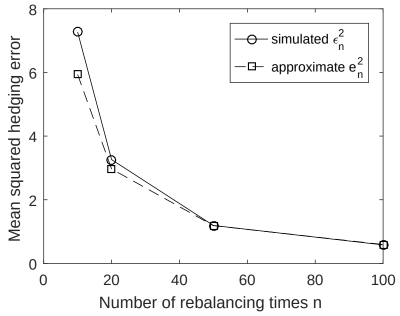
(a) $g(t,\chi)$. **(b)** $v(0,S,A)$.

Figure 4: Fixed strike call: (a) Soluiton $g(t,\chi)$ of the reduced equation (27) and (b) function $v(0,S,A)$ describing the MSHE.

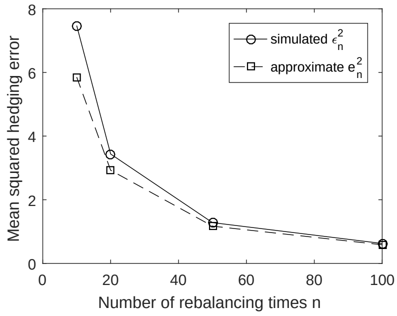
Panel (a) of Figure 4 shows the solution $g(t,\chi)$ of (27) for a fixed strike call with $r=0.15$, $\sigma=0.30$, $\hat{\delta}=0$, $T=1$, $\mu=0.2$ and $K=100$. Panel (b) of the same figure shows the corresponding solution $v(t,S,A)$ of (24) at time $t=0$.

3.4.3 Mean squared hedging error approximation

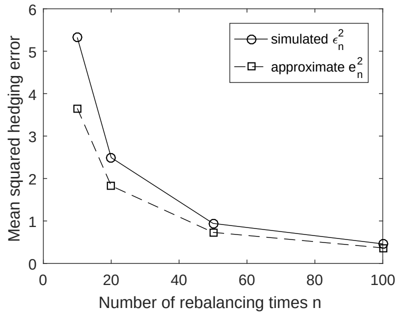
To verify the validity of approximation (30) we compared it to actual mean squared hedging errors obtained by Monte Carlo simulations.



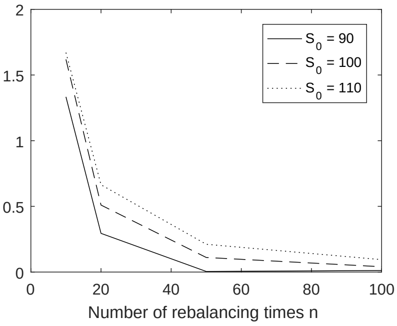
(a) $S_0 = 90$.



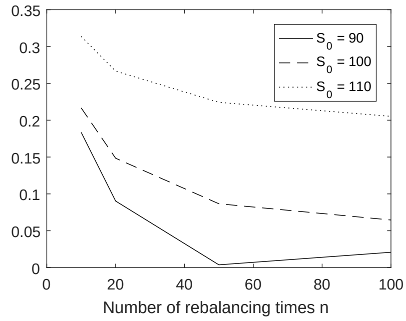
(b) $S_0 = 100$.



(c) $S_0 = 110$.



(d) $|\varepsilon_n^2 - e_n^2|$.



(e) $|\varepsilon_n^2 - e_n^2| / \varepsilon_n^2$.

Figure 5: Fixed strike call: Top row shows the development of ε_n^2 and e_n^2 for increasing number of trading times and different S_0. Bottom row shows the (d) absolute and (e) relative differences between the simulated and approximate MSHE.

Top row of Figure 5 confirms that the MSHE tends to zero as trading becomes more frequent and also that the approximate MSHE e_n^2 agrees with the simulated MSHE ε_n^2. The absolute differences of ε_n^2 and e_n^2, shown in panel (d), get close to zero with more frequent rebalancing and the relative differences, shown in panel (e), get to around 5% except the case with $S_0 = 110$ where the difference is 20.5%. However, the difference does have a decreasing tendency as can be seen in panels (c) and (e) of Figure 5.

3.5 Conclusion

In the second part of the thesis we focused on the mean squared hedging error of a discretely implemented delta hedging strategy for Asian options. We heuristically derived the MSHE approximation (21) which is consistent with literature. We applied this approximation to the Asian option case to obtain (22) and proposed that it can be evaluated by solving the system of PDEs (23), (24). Furthermore, we showed that dimension of the system can be reduced to (26), (27).

We numerically solved the reduced system (26), (27) and used the obtained solution $g(t, \chi)$ to evaluate the Asian option MSHE approximation by (30). We compared this approximation to actual MSHE obtained by simulations to find that the approximation fits reasonably well with the actual mean squared hedging error.

4 Conclusion

In this dissertation thesis we examined two applications of optimization in financial mathematics. The first part dealt with optimal liquidation when the selling price is adversely affected by the current liquidation rate. Our formulation differs from most of optimal liquidation literature in giving the pressure to liquidate endogenously and using a stochastic time horizon which is determined as a part of the optimal strategy. The problem reduces to solving a severely singular initial value problem IVP_0 for ODE (6), as shown in [4]. We proposed a method of overcoming the singularity which leads to numerically stable solutions and we also confirmed that the found solution is indeed the value function of the optimal liquidation problem. The presented research in this field could be extended to in the future by including permanent price impact in the model or by considering other than linear utility functions.

The second part of the thesis mean squared hedging error of discrete delta hedging strategies for Asian options in context of quadratic hedging. We heuristically derived an approximation of the MSHE, which is consistent with known approximations from literature, and applied it to Asian options. Numerically, we confirmed that the approximation performs well when confronted with actual MSHE obtained from Monte Carlo simulations. An obvious extension of the research presented in this part of the thesis is proving rigorously that the heuristically derived MSHE approximation (21) indeed holds. Other possible extensions include using generalized geometric Brownian motion or Lévy models for the option price or including transaction costs.

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Výučba

- FINAČNÁ MATEMATIKA, vedenie cvičení pre tretí ročník bakalárskeho odboru Manažérska matematika, zimný semester 2014-2017.
- METÓDY VOĽNEJ OPTIMALIZÁCIE, vedenie cvičení pre druhý ročník bakalárskeho odboru Ekonomická a finančná matematika, letný semester 2015–2018.

Vedenie záverečných prác

Diplomové práce

- Tomáš BUČEK: *Oceňovanie opcií závislých od cesty pomocou Monte Carlo metód*, 2018.

Bakalárske práce

- Jakub HRBÁŇ: *Oceňovanie opcií pomocou neurónových sietí*, 2018.
- Samuel HORVÁTH: *Ilustrácia optimalizácie portfólia pomocou Shiny*, 2017.
- Andrej REMÁK: *Binomické metódy oceňovania ázijských opcií*, 2017.
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