Univerzita Komenského v Bratislave

Fakulta matematiky, fyziky a informatiky

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Autoreferát dizertačnej práce

Dynamic Stochastic Accumulation Model with Application to Portfolio Risk Management

na získanie akademického titulu *philosophiae doctor*

v odbore doktorandského štúdia:
9.1.9 Aplikovaná matematika

Bratislava 2014

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na katedre aplikovanej matematiky a štatistiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave

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1 Introduction

One of the most important problems faced by investors involve the allocation of their wealth among different investment opportunities in a market consisting of risky assets. Determination of optimal portfolios is a rather complex problem depending on the objective of the investor. The problem of optimizing the portfolio returns has been discussed by many authors and the voluminous literature is devoted to solve such problem. The most known solution for portfolio optimization has been introduced by Markowitz (1952) and belongs to static portfolio optimization methods. The task to estimate the evolution on the financial market has shown to be very difficult. Especially in the recent years the financial markets are characterized by high volatility which causes extreme values of asset prices and their returns. The distributions of many financial quantities were shown to have heavy tails and exhibit skewness and other non Gaussian properties. These observations led to the new approach of modeling the asset returns by means of fat tailed probability distribution.

In the dissertation thesis we study the impact of the choice of fat-tailed distribution of asset returns when optimizing portfolio. We consider stochastic accumulation model in application saving management discussed by Kilianová, Melicherčík and Ševčovič in series of articles, see [3], [4], [6], [7]. We replace the modeling of returns by the normal distribution with non symmetrical distribution typical for their fatter tails.

2 Goals of the thesis

The main goal of the thesis is to study and implement the fat tailed distribution of the returns into financial planning problems. We adopt the dynamic accumulation model originally designed for the optimal saving management and consider fat tailed Normal-Inverse-Gaussian distribution for the asset returns and the portfolio return. We aim to analyze the pitfalls arising from considering the fat tailed distribution characterized by exhibiting large kurtosis and introduce a stable numerical method for solving formulated optimization problem.

We study the distribution of the final wealth and its properties for normal and for NIG distribution. Our following aim is to provide a sensitivity analysis of the model to the model parameters, especially to the parameters of the portfolio return distribution. We focus on the dependency of final accumulated sum on portfolio descriptive statistics and study the influence of a small change in one input parameters to final wealth descriptive statistics and optimal choice.

3 Model formulation

The model describes the problem of an investor starting from initial capital with infinite time horizon. A special case is related to pension fund management for clients who contribute regularly to selected funds in order to maximize their pension after certain time horizon, at the retirement time.

The model is based on the recovery of initial capital with the possibility of rebalancing at each time step. We assume an investor having information about the history of the market denoted by I_t for each investment time $t \in \mathbb{R}^+$, possessing capital of C_0 units, which he can invest in a finite set of investment opportunities, set of N different assets. Let X_t for $t \in \mathbb{R}^+$ denote the capital of the investor at the beginning of the time period. Then the investor's capital is evolving in time according to equation

$$X_{t+\Delta t} = X_t e^{r_t^\theta \Delta t}, \tag{3.1}$$
$$X_0 = C_0, \tag{3.2}$$

where r_t^θ represents the return of the portfolio composed from opportunities $i = 1, ..., N$ with weights θ_i.

3.1 Model with regular contributions

The problem is defined as a finite horizon problem with horizon T and with portfolio containing only long positions in assets from finite set of investment opportunities. We consider an investor possessing an initial capital C_0 and contributing regularly the value of value $C\Delta t$ in times corresponding to rebalance time points (the yearly contribution corresponds to the value C). The wealth evolution X_t can be described as following

$$\begin{aligned} X_{t+\Delta t} &= X_t e^{(r_t^{\theta_t} - \rho_t)\Delta t} + C\Delta t, \\ &\equiv F_t(X_t, \theta_t, r_t^{\theta_t}), \quad t \in [0, T), \\ X_0 &= C_0, \end{aligned} \tag{3.3}$$

where r_t^θ represents the return of portfolio with weights θ_t in the time period $[t, t + \Delta t)$. The portfolio can consist from risky, low-risky and non-risky assets. The portfolio weights fulfill the condition $\sum_i \theta_t^i = 1$ for each t, i.e. the risk free assets are considered to be part of the portfolio with volatility close to zero. The $\rho_t > 0$ corresponds e.g. to inflation rate.

3.2 Pension problem

Pension management problem is a special case of the model with regular contribution adapted to the needs of pension planning. The pension problem is a long-term horizon problem bound to the client's salary, economic prediction for the salary growth and restrictions introduced by Government. The pension problem suited for Slovak pension system has been discussed by Ševčovič, Melicherčík and Kilianová.

We suppose that the future pensioner deposits once a year a τ-part of his yearly salary w_t in a pension fund with composition expressed by θ with respect to the Governmental restriction. After the retirement time T the pensioner usually strives to maintain his living standard. Therefore the accumulated saved amount, from the pensioner's point of view, is not really what he is interested in. The ratio of the cumulative sum s_T and the yearly salary w_T, $d_T = \dfrac{s_T}{w_T}$, is more important. Using the quantity $d_t = \dfrac{s_t}{w_t}$ the budget-constraint equations can be formulated as

$$\begin{aligned} d_{t+\Delta t} &= d_t e^{(r_t^{\theta_t} - \rho_t)\Delta t} + \tau\Delta t, \\ &\equiv F_t(d_t, \theta, r), \quad t \in [0, T), \\ d_0 &= \tau\Delta t, \end{aligned} \tag{3.4}$$

where $r_t^{\theta_t}$ is the return of the fund with portfolio composition θ_t in time period $[t, t+1)$, ρ_t denotes the wage growth in $[t, t+1)$ and T is the expected retirement time, [3]. The salary of the saver follows a deterministic process given by equation $w_{t+\Delta t} = w_t e^{\rho_t \Delta t}$. We assume that the term structure of wage growth is known and can be estimated by means of an econometric model.

3.3 Problem formulation

Investor's satisfaction and risk attitude are often described by the utility function. The set of investor's opportunities is created by assets behaving in stochastic manner. We suppose that the investor's utility U with the risk aversion coefficient is known as well as the asset distribution.

The choice of the opportunity i from the set of N assets depends on the investor's profile and his attitude to risk. Intuitively, the investor's risk aversion decreases with his wealth. Therefore, we formulate the problem in the way: *at given level of investor's risk aversion we maximize the expected utility from his wealth at time T.*

The investor decides at every time step t for portfolio composition θ_t according to information I_t containing the history of all asset returns. We formulate the problem as stochastic dynamic problem

$$\max_{\theta \in \Theta} \mathbb{E}(U(X_T)) \tag{3.5}$$

subject to budget constraint given by (3.3).

We apply a fact from the theory of conditional expectations that a sequence of non-decreasing information $\{I_t, t \in [0, T)\}$ may be considered as a sequence of non-decreasing σ-algebras. This allows the implementation of the tower law for conditional expectations on (3.5).

Denoting the investor's intermediate utility function at time t as $V_t(X) = \max_{\theta \in \Theta} \mathbb{E}(U(X_T)| X_t = X)$ and applying the tower law, we obtain the Bellman equation

$$\begin{aligned} V_t(X) &= \max_{\theta} \mathbb{E}(V_{t+\Delta t}(F_t(X, \theta, r))) \\ V_T(X) &= U(X). \end{aligned} \tag{3.6}$$

The solution of (3.6) gives to the investor the information about the optimal portfolio composition θ in every time t in dependency on the random variable representing current wealth X_t. Supposing that the compound probability distribution of each portfolio composition is known and is represented by density f_t^θ, the equation (3.6) can be rewritten into form

$$\begin{aligned} V_t(X) &= \max_{\theta} \mathbb{E}(V_{t+\Delta t}(F_t(X, \theta, r))) \\ &= \max_{\theta} \int_{\mathbb{R}} V_{t+\Delta t}(F_t(X, \theta, r)) f_t^\theta(r) \mathrm{d}r. \end{aligned} \tag{3.7}$$

According to Proposition 3.1 the optimal solution exists and is unique for increasing function F_t in X. The solution denotes the decisions process for the optimal opportunity choice in each decision time t.

Proposition 3.1. *Let $U(X)$ be an increasing, strictly concave, $\mathcal{C}^2$ smooth function for $X > 0$. Then for any $t = 0, \Delta t, 2\Delta t \dots, T - \Delta t$,*

1. *the function $V_t(X)$ is increasing and strictly concave in X-variable;*
2. *there exists the unique argument $\hat{\theta}_t(X)$ of the maximum in (3.7).*

3.4 Utility function

Usual assumption is that the function $U(x)$ is twice differentiable; with (i) $U'(x) > 0$ and (ii) $U''(x) < 0$. The first property amounts to the evident requirement that more is better. The U' is referred to as a marginal utility.The second property is referred to as a risk aversion.

We suppose the utility functions of CRRA type. The first CRRA power utility function with coefficient $\alpha > 0$ representing the investor's risk aversion is given by

$$U(x) = \frac{x^{1-\alpha}}{1-\alpha}. \tag{3.8}$$

The model formulation (3.3) for $C_0 = C\Delta t$ allows to derive an interesting property that the level of wealth is proportional to the contribution rate, i.e.

$$\mathbb{E}(X_t^{\lambda C}) = \lambda \mathbb{E}(X_t^C), \tag{3.9}$$

for any t and $\alpha > 0$. Process X_t^C is evolving according to (3.3) with regular yearly contribution of value C.

4 Normal inverse Gaussian distribution

Portfolio optimization is based on trading of risk and return. The construction of portfolios with minimum risk for a given return depends on two inputs: the choice of the risk measure, and the probability distribution used to model returns. Although the normal distribution for modeling returns was widely used for many years, we can find a voluminous literature concerning modeling returns with probability distribution which may better take in account more extreme changes. It can be shown that for many assets the returns do not behave "normally". It has been observed that the fat tails are one of the features of the asset returns distribution. Andersen, Bollerslev, Diebold, Ebens in [1] show that the daily DJIA (Dow Jones Industrial Average) returns, have fatter tails than the normal and, for the majority of the stocks, are also skewed.

The NIG distribution is a special case of generalized hyperbolic distribution. It has four parameters $\alpha, \beta, \mu, \delta$ specifying the shape of the density function. Barndorff-Nielsen [2] defined the NIG distribution as a normal variance-mean mixtures when the mixture distribution is a inverse Gaussian distribution.

Definition 1. *The random variable X is normal inverse Gaussian distributed $NIG(\alpha, \beta, \mu, \delta)$ if its probability density function is given by*

$$f(x) = \frac{\alpha}{\pi} \exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\} \frac{K_1(\alpha\delta\sqrt{1 + (\frac{x-\mu}{\delta})^2})}{\sqrt{1 + (\frac{x-\mu}{\delta})^2}} \tag{4.1}$$

where K_1 denotes the modified Bessel function of the third kind, and the conditions for the parameters are $\alpha > 0, \delta > 0, \mu \in \mathbb{R}, 0 \leq |\beta| \leq \alpha$.

Normal inverse Gaussian distribution exhibits properties stated in Proposition 4.1.

Proposition 4.1. *The NIG class of densities has the following properties:*

1. *Scaling property: If $X \sim NIG(\alpha, \beta, \mu, \delta)$, then $Y = cX \sim NIG(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta)$.*
2. *Convolution property: If $X_1 \sim NIG(\alpha, \beta, \mu_1, \delta_1)$ and $X_2 \sim NIG(\alpha, \beta, \mu_2, \delta_2)$ are independent, then the sum $Y = X_1 + X_2 \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$.*
3. *Standardization: If $X \sim NIG(\alpha, \beta, \mu, \delta)$, then variable $Y = \dfrac{X - \mu}{\delta}$ has the Standard Normal Inverse Gaussian Distribution $NIG(\alpha\delta, \beta\delta, 0, 1)$.*

The first four moment mean, variance, skewness and kurtosis can be expressed by means of NIG parameters. Similarly, the NIG parameters can be expressed using the moments as stated in Theorem 4.1.

Theorem 4.1. *Suppose that random variable X is $NIG(\alpha, \beta, \mu, \delta)$ distributed and its mean, variance, skewness and excess kurtosis are denoted as $\mathbb{E}, \mathcal{V}, \mathcal{S}$ and $e\mathcal{K}$, respectively. Then the parameters are related to the moments by*

$$\alpha = \frac{3}{\sqrt{\mathcal{V}}} \frac{(3e\mathcal{K} - 4\mathcal{S}^2)^{\frac{1}{2}}}{(3e\mathcal{K} - 5\mathcal{S}^2)} \tag{4.2}$$

$$\beta = \alpha \frac{\mathcal{S}}{(3e\mathcal{K} - 4\mathcal{S}^2)^{\frac{1}{2}}} \tag{4.3}$$

$$\delta = \alpha \mathcal{V} \left(1 - \frac{\beta^2}{\alpha^2}\right)^{\frac{3}{2}} \tag{4.4}$$

$$\mu = \mathbb{E} - \delta \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \tag{4.5}$$

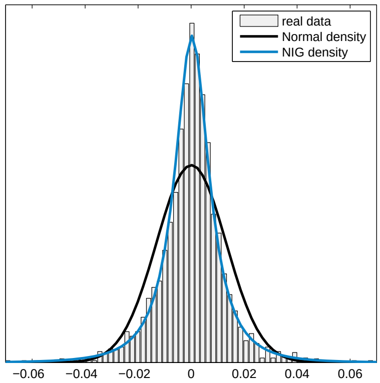
under condition that $3e\mathcal{K} - 5\mathcal{S}^2 > 0$.

4.1 Motivation for using normal inverse Gaussian distribution

Onalan in his article [8] provided an empirical analysis of financial data, especially he focused on S&P Index and VIX Index. He investigated the use of normal inverse Gaussian distribution in financial risk management. We have adopted the NIG model and analyze the fitness on the real data. In Figure 4.1 we offer an illustration how the real data are actually fitted by the normal distribution and the fat tailed distribution NIG. We have considered the daily data for S&P500 Index in time period 02/01/2000 - 02/01/2014.

4.2 Drawbacks of NIG distribution

NIG distribution has the great property to describe the density shape of the variable in very good way and depict its skewness and kurtosis. The replication is straightforward from NIG distribution with parameters obtained by Theorem 4.1.



Mean (p.a.)	1.65%
St. deviation (p.a.)	20.87%
Skewness	-0.17
Kurtosis	7.7

Figure 4.1 Histogram of log returns for daily data for S&P Index and density of probability distribution for time period Jan 2000 - Jan 2014.

One drawback follows directly from NIG convolution property (Theorem 4.1) defined only for independent variables exhibiting same shape parameters which is a very restrictive condition. Considering more dimensional space of uncorrelated variables leads to non-problematic replication exhibiting the pre-desribed values of mean, variance, skewness and kurtosis. The problem appears when considering more dimensional space of correlated NIG distributed variables. *The higher the correlation and the higher the skewness and kurtosis of the variables, the less precise replication.*

Therefore we introduce the algorithms for generating mixtures created by correlated NIG distributed variables.

5 Mixture of NIG distributed time series

The normal inverse Gaussian distribution does not belong to class of infinitely divisible distributions, i.e. the mixture of two or more NIG distributed variables does not have to create NIG distributed variable unless the mixing variables share the shape parameters α and β. This feature of NIG distribution causes difficulties when assuming a portfolio composed from assets with NIG distributed returns. In general, it is not usual that more assets follow distributions with same shape parameters. Therefore, we introduce a procedure for approximation of parameters of NIG mixtures.

Another issue arising by creating NIG mixtures follows from the properties of the historical time series often exhibiting high correlation and their modeling (replication) requires to take this correlation into consideration. In the next sub-sections we offer two algorithms for replicating the time series with pre-described four moments and correlation.

5.1 Replication of time series: Choleski approach

Replication of the time series from pre-defined moments and correlation create an important part by generating the NIG mixtures. We first analyze an algorithm based on

Choleski decomposition commonly used by generating correlated normal random variables. Despite of the difference between the NIG and normal distribution the algorithm gives for some set of parameters sufficiently precise results.

However, if the time series are correlated, the replicated times series do not have to exhibit the pre-described values of the four moments and correlation. We have provided an analysis regarding the correctness of replication of asset returns when considering mixture of two correlated NIG distributed variables and state our results in Table 5.1.

Table 5.1 Table of observed dependencies of pre-described and replicated time series properties by Choleski decomposition based algorithm.

Skewness	Kurtosis	Corr $\lvert\rho\rvert$	Consequence
$\mathcal{S}(A_1) = 0$ $\mathcal{S}(A_2) = 0$	$\mathcal{K}(A_1) = \mathcal{K}(A_2)$	low	✓
		higher	the higher correlation causes worse fit of $\mathcal{K}(A2)$, higher $\mathcal{K}$ worse $\mathcal{K}(A_2)$
$\mathcal{S}(A_1) \neq 0$	$\mathcal{K}(A_1) = \mathcal{K}(A_2)$	low	the higher the skewness of asset 2 $\lvert\mathcal{S}(A_2)\rvert$, the worse correlation
$\mathcal{S}(A_2) \neq 0$		higher	the higher correlation $\lvert\rho\rvert$, the worse $\mathcal{S}(A_2)$ (and $\lvert\mathcal{K}(A_2)\rvert$)
$\mathcal{S}(A_1) = 0$	$\mathcal{K}(A_1) \neq \mathcal{K}(A_2)$	low	correlation does not correspond to pre-described correlation ($\approx \rho/2$)
$\mathcal{S}(A_2) = 0$		higher	worse correlation and worse $\mathcal{K}(A_2)$
$\mathcal{S}(A_1) \neq 0$	$\mathcal{K}(A_1) \neq \mathcal{K}(A_2)$	low	correlation does not correspond to pre-described correlation
$\mathcal{S}(A_2) \neq 0$		higher	worse correlation and worse $\mathcal{K}(A_2)$

5.2 Replication of time series: Optimization approach

We assume a NIG mixture consisting from two assets, whose properties are given by their first four moments and correlation. The aim is to generate the time series satisfying these inputs. However, the replicated time series do not maintain the input properties and thus the time series have to be generated by means of other parameters. The objective is to find the set of parameters, that would replicate time series exhibiting pre-described properties.

Since the transformation of the independent random variables to correlated variables with given correlation by means of Choleski decomposition keeps the first variable unchanged, we fix the realization of the first asset while the moments of the second asset and the correlation will be a subject of optimization. The algorithm searches a set of 4 parameter - variance, skewness and kurtosis for the second asset and correlation of asset 1 and asset 2 using the genetic algorithm, that would replicate by using the Choleski approach time series with predefined moments and correlation.

Remark 5.1. *The use of the optimization based algorithm seem to be effective for time series with relatively high correlation ($\lvert\rho\rvert > 0.4$) and different kurtosis. The effectiveness of use of the genetic algorithm vanishes with decreasing correlation.*

For $|\rho|$ low the optimization based algorithm works effectively (sufficient replication properties and short time) if the optimizing parameter is only correlation ρ and other model parameters are fixed.

5.3 Generating NIG mixtures with predefined moments

We consider a portfolio consisting from N assets with NIG distributed returns, $i = 1, \ldots, N$, with weights θ, where $\theta = (\theta_1, \ldots, \theta_N)^T$. We presume that the portfolio returns are also NIG distributed, i.e. $r_P^\theta \sim NIG(\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta)$. To estimate the parameters $\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta$ of the portfolio returns, we first define an approximation of a NIG mixture.

Definition 2 (NIG approximation of mixture)**.** *Assume that $r_i \sim NIG(\alpha_i, \beta_i, \mu_i, \delta_i)$ for $i = 1, 2, \ldots, N$. Then for weighted mixture $r = \sum_{i=1}^N \theta_i r_i$ the aim is to find parameters $\alpha^\theta, \beta^\theta, \mu^\theta, \delta^\theta$ of NIG distribution dependent on vector θ such that it holds*

1. *for $\theta = \mathbf{e}$ the parameters agree exactly ($\mathbf{e}$ is a unit vector);*
2. *first four parameters of convex combination $r = \sum_{i=1}^N \theta_i r_i$ agree with four moments of r for each $\theta_i \in [0, 1], \sum_i \theta_i = 1$.*

Assumption 5.1. *We assume that for each asset i, $i = 1, \ldots, N$ the values of first four moments of asset returns ($\mathbb{E}_i, \mathcal{V}_i, \mathcal{S}_i, e\mathcal{K}_i$) and the corresponding correlation matrix Σ are known or can be calculated from real time series.*

The algorithm is presented in Table 5.2.

Table 5.2 NIG MIXTURE ALGORITHM

```
INPUT

    Return of asset i (A_i): E_i = E(A_i), V_i = V(A_i), S_i = S(A_i), K_i = K(A_i);

    correlation matrix Σ = corr(i,j)  for i,j = 1,...,N.

ALGORITHM

  1. replicate the asset returns X_i for i = 1,...,N by Choleski
     decomposition based algorithm or by Optimization based
     algorithm;

  2. compute the portfolio return as linear combination of asset
     returns weighted according to composition θ, i.e. r^θ =
     Σ_{i=1}^N θ_i X_i;

  3. compute the values of the first four moments E(r^θ), V(r^θ), S(r^θ) and
     K(r^θ) for the vector r^θ;

  4. calculate NIG parameters.

     Remark: It is assumed that the random vector r^θ representing the
     portfolio returns is also NIG distributed.
```

6 Numerical approximation of stochastic dynamic optimization problem

In this section we describe the numerical approximation procedure for solving the formulated maximization problem (3.5) with dynamic constraints given by (3.3). The numerical scheme for solving this problem has been described in details by Kilianová, Melicherčík and Ševčovič, e.g. in [4]. The proposed scheme has been constructed for needs of implementation of normal distribution. The NIG distribution exhibits a specific shape and requires also a special treatment for its numerical approximation. We describe the numerical process with a focus on needs of use of NIG distribution or generally distribution with fat tails and high kurtosis.

The problem is formulated as a decision problem. In each time t the investor decides for the stock-to-bond proportion θ according to the problem given by Bellman equation. The problem to be solved in each time step t is to find the maximum over θ of the integral

$$\int_{\mathbb{R}} V_{t+\Delta t}(x_{t+\Delta t}) f^{\theta}(r) \mathrm{d}r. \tag{6.1}$$

The values of V_t are calculated in space points $\{x_i, i = 1, ..., n_x\}$ with $x_1 = x_{\min}$ and $x_{n_x} = x_{\max}$ (equidistant) and in time points corresponding to Δt. In each grid point the values are calculated for $\{\theta_j, j = 1, ..., n_\theta\}$ of the interval $[0, 1]$ (equidistant) where n_θ is sufficiently large.

The density function f^θ is considered to be the density function of the portfolio returns. When considering the NIG density function f^θ the NIG parameters $\alpha^\theta, \beta^\theta, \mu^\theta$ and δ^θ are as discussed in section 5. For normal distribution the formula for parameters μ^θ and σ^θ is known and no approximation algorithm is needed.

The peak of the NIG density function can be due to the high kurtosis thin and high. The classical trapezoidal rule with uniform spacing grid for approximating the definite integral is not sufficient, since the grid should be denser in neighborhood of the center of the peak. For the Bellman type integral with NIG distribution densities f^θ we therefore use the Legendre-Gauss quadrature with $n > 0$ nodes. The quadrature rule is based on approximation of the integral by the weighted sum of function values at specified points within the domain of integration. We consider interval I_r to be sufficiently large to cover all non zero values, $I_r = (\overline{r^\theta} - 15\sigma^\theta, \overline{r^\theta} + 15\sigma^\theta)$, using $n = 50$ nodes. The efficient interval I_r for normal distribution or distributions with thinner tails can be much smaller.

As far as the values of function V_t are computed only in discrete points x_i, we need to define a proper approximation of the values $\xi \in [x_{\min}, x_{\max}]$ as well as for values outside of the defined interval. Especially, the approximation of $V_t(\xi)$ in points outside the space grid $\xi > x_{\max}$ require a special attention. We define the approximation of $V_t(\xi), \forall \xi \in [0, \infty]$ as following

1. for $\xi < x_{\min}$ we set $V_t(\xi) = V_t(x_{\min})$ (this restriction can be viewed as the bottom value that has to be ensured);

2. for $\xi > x_{\max}$, $V_t(\xi)$ is set as (6.3);

3. for $\xi \in [x_{\min}, x_{\max}]$ an interpolation of nearest neighboring grid points is calculated as (6.4).

6.1 Boundary conditions

The correct setting of the boundary condition creates a crucial part of the numerical solution. The error caused by incorrect valuation of the value function outside the grid accumulates by backward calculation and thus strongly influences the solution. In the portfolio optimization problem the contour lines of optimal choice θ could be therefore incorrectly curved.

The boundary condition for $\xi > x_{\max}$ set as $V_t(\xi) = V_t(x_{\max})$ would require a large value of $x_{\max}$ which would be very time consuming since the division of the interval has to be dense. Another option is to use an extrapolation for approximation of $V_t(\xi)$. However, a decline in the rate of growth of the value function V_t (can be seen already for $V_T(.) = U(.)$) causes that the values obtained by extrapolation are strongly deviated from the true values. The error accumulates therefore from the first step, for $t = T$. These facts leaded to derivation of the heuristic of a boundary condition specific for our type of problem.

We propose 6.1 assumed by deriving the heuristic for boundary condition specific for our model.

Lemma 6.1. *For utility function in form (3.8) and wealth evolution given by (3.3), for $x \to \infty$ the integral can be approximated as*

$$\lim_{x\to\infty} \int_{\mathbb{R}} \left(\frac{xe^{(r^\theta-\rho)\Delta t} + C\Delta t}{x} \right)^{1-\alpha} f(r)\mathrm{d}r = \int_{\mathbb{R}} e^{(r^\theta-\rho)\Delta t(1-\alpha)} f(r)\mathrm{d}r. \tag{6.2}$$

For model without the contributions, i.e. $C = 0$, the solution is exact.

We approximate the value of $V_t(\xi)$ for $\xi > x_{\max}$ as

$$\begin{aligned} V_t(\xi) &= \beta_t U(\xi) \\ \beta_t &= \gamma_T \gamma_{T-\Delta t} \cdots \gamma_{t+\Delta t} \beta_T \\ \gamma_t &= \min_\theta \int_{\mathbb{R}} e^{(r^\theta-\rho_t)\Delta t(1-\alpha)} f^\theta(r)\mathrm{d}r. \end{aligned} \tag{6.3}$$

6.2 Interpolation of the value function

The value function of the point not lying on the grid has to be approximated by means of the adjacent grid points. The value of V_t for any $\xi \in [x_{\min}, x_{\max}]$ can be expressed by means of the interpolation of values $V_t(x_i)$ and $V_t(x_{i+1})$ corresponding to the closest grid points fulfilling $\xi \in [x_i, x_{i+1}]$.

We offer option based on copying the shape of the utility function and its mapping to calculated values. We use the shape of the utility function and compute $V_t(\xi)$ as

$$\begin{aligned} V_t(\xi) &= c_\xi U(\xi), \\ c_\xi &\approx c_{x_i} + \frac{c_{x_{i+1}} - c_{x_i}}{x_{i+1} - x_i}(\xi - x_i), \end{aligned} \tag{6.4}$$

where c_ξ is calculated as a linear interpolation of ratios of V_t and U scaled according to β_t, i.e. $c_{x_i} = \dfrac{V_t(x_i)}{\beta_t U(x_i)}$ and $c_{x_{i+1}} = \dfrac{V_t(x_{i+1})}{\beta_t U(x_{i+1})}$. This approach enables the use of lower space division n_x than by the linear interpolation while achieving the same result.

Remark 6.1. *Since the values inside and outside the space grid point should create a continuous function, the values $V_t(\xi)$ for $\xi > x_{\max}$ have to be parallel shifted such that the value $V_t(x_{\max})$ equals to $V_t(x_{\max})$ obtained by (6.3).*

7 Results

In this chapter we examine the proposed numerical scheme on the saving management problem designed for the II. pillar of the Slovak pension system. We follow the discussed dynamic model and maximize the expected utility of the final accumulated wealth of the pensioner.

We apply the algorithm for normally and NIG distributed returns and compare the trajectories of the expected wealth evolution during the investment time and specially the optimal decision for portfolio composition for normal and for NIG distributed portfolio returns. We analyze the distribution of the final wealth and its properties and discuss the impact of the considered skewness and kurtosis of the asset returns.

7.1 Parameters

We consider a Slovak future pensioner whose retirement time T is 40 years. According to Slovak pension system, he contributes into his saving account in II. pillar very month 8.91% of his yearly salary. Further we assume that the pension management institutions invest only into two assets. Stocks are represented by S&P500 Index and bonds by ten years US governmental bonds. In each fund there are restriction given by the Slovak Government in Equation 7.1. The wage growth ρ_t in Slovakia was taken from a paper by Kvetan et al. [5]. We further use a grid for numerical calculation with $\Delta t = 1$ (rebalancing is possible once a year), equidistant space division of interval $[0.01, 50]$ with $n_x = 1200$ and equidistant division of $\theta \in [0, 1]$ with $n_\theta = 100$.

$$\theta_t = \begin{cases} [0, 0.8] & \text{if } T - t > 15 \text{ (last 15 years of saving),} \\ [0, 0.5] & \text{if } T - t > 7 \text{ (last 7 years of saving),} \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

We consider the same time period Jan 1996 - Jun 2002 as in Kilianová et al. [4] for time series representing the portfolio assets. The bond yield as a non-risky asset is characterized by small volatility which is 0.82% with expected return 5.16% per year. On the other hand, the risky-assets offer higher yield but under higher risk. The S&P 500 Index in considered time period yields to 10.28% per year with volatility almost 16.9%. The correlation of their returns is -0.1151. We calculate the second two moments for needs of NIG distribution, i.e. skewness and kurtosis of the bond are -0.05 and 3.6, respectively and for stock -0.24 and 5.92, respectively.

7.2 Comparison: Normal vs. NIG distribution

The optimal choice for portfolio composition of the future pensioner has the characteristics that in the early years, he prefers a high proportion of risky assets which decreases with shortening of time to retirement. The pensioner tends to decide for more conservative portfolios in the last years of savings. The preferences change also in dependence on the current accumulated wealth, i.e the higher accumulated sum the lower proportion of the risky assets in portfolio. Due to the Governmental restrictions, the optimal choice is regulated and the decision might be strongly affected.

With increasing risk aversion, the investor tends to take more conservative decisions for same time and level of accumulated sum. The Figure 7.1 illustrates the trajectory of the optimal choice of the saver for tree different coefficient levels $\alpha = 5, 9, 13$. By both cases, with (a) and without (b) the regulations we observe that the saver tends to reduce the risky part of the portfolio sooner for higher α. The proportion of stocks reduces with time for all α. The same behavior can be observed by considering NIG distribution, however saver tends to decide for more risky portfolios in the first years and then fast change the portfolio into more conservative as it can be seen from Figure 7.2.

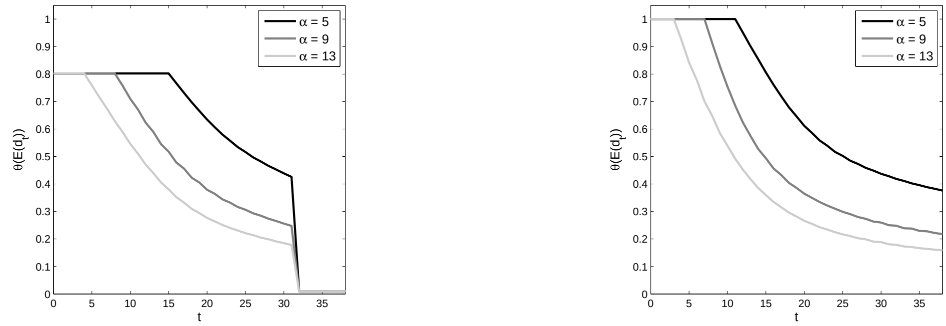


Figure 7.1 Evolution of optimal choice θ for different levels of risk aversion coefficient during the saving period considering normal distribution with (on the left) and without applying regulations (on the right).

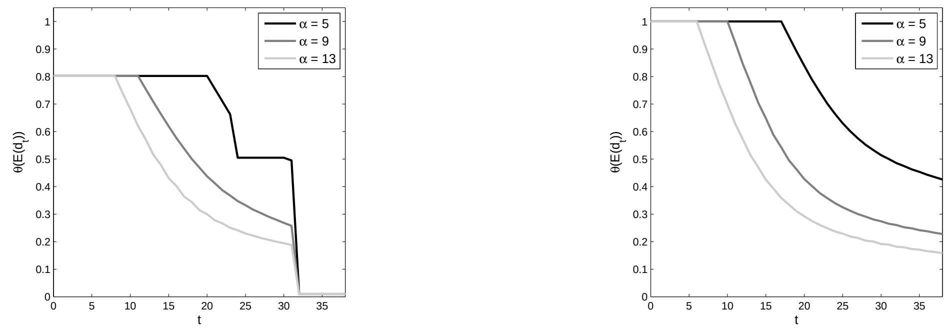


Figure 7.2 Evolution of optimal choice θ for different levels of risk aversion coefficient during the saving period considering NIG distribution with (on the left) and without applying regulations (on the right).

Smaller α indicates riskier portfolio and thus implies a higher expected value of the final wealth $\mathbb{E}(d_t)$. The mean value $\mathbb{E}(d_T)$ is always higher when there are no governmental limits. The shape of the empirical distribution of the final wealth d_T for high

Table 7.1 Properties of the final wealth distribution for different levels of risk aversion coefficient considering normal distribution with (on the left) and without applying regulations (on the right).

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	5.17	4.44	4.11
$\mathcal{S}td(d_T)$	1.55	0.81	0.52
$\mathcal{S}(d_T)$	1.03	0.57	0.42
$\mathcal{K}(d_T)$	4.86	3.45	3.28
$Q_{10\%}(d_T)$	3.43	3.47	3.47
$Q_{50\%}(d_T)$	4.92	4.36	4.07
$Q_{90\%}(d_T)$	7.23	5.50	4.79

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	6.13	4.85	4.34
$\mathcal{S}td(d_T)$	2.22	1.02	0.63
$\mathcal{S}(d_T)$	1.23	0.62	0.48
$\mathcal{K}(d_T)$	5.79	3.65	3.44
$Q_{10\%}(d_T)$	3.69	3.63	3.57
$Q_{50\%}(d_T)$	5.75	4.74	4.30
$Q_{90\%}(d_T)$	8.98	6.19	5.18

α is characterized by higher and sharper peak implying higher concentration of the observations around the median. The standard deviation is therefore lower as well as the skewness. On the hand, the 10% quantile is comparable for all α coefficients, however the median and 90% quantile are much higher for lower α. The statistics of the final wealth for normal distribution are summarized in Table 7.1 and for NIG distribution in Table 7.2.

Table 7.2 Properties of the final wealth distribution for different levels of risk aversion coefficient considering NIG distribution with (on the left) and without applying regulations (on the right).

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	6.13	4.87	4.35
$\mathcal{S}td(d_T)$	2.28	1.01	0.62
$\mathcal{S}(d_T)$	1.30	0.65	0.44
$\mathcal{K}(d_T)$	5.91	3.89	3.57
$Q_{10\%}(d_T)$	3.68	3.65	3.61
$Q_{50\%}(d_T)$	5.70	4.77	4.31
$Q_{90\%}(d_T)$	9.15	6.18	5.16

	$\alpha = 5$	$\alpha = 9$	$\alpha = 13$
$\mathbb{E}(d_T)$	8.19	5.47	4.67
$\mathcal{S}td(d_T)$	3.62	1.27	0.75
$\mathcal{S}(d_T)$	1.37	0.67	0.48
$\mathcal{K}(d_T)$	6.50	3.92	3.48
$Q_{10\%}(d_T)$	4.31	3.96	3.75
$Q_{50\%}(d_T)$	7.51	5.34	4.60
$Q_{90\%}(d_T)$	12.90	7.11	5.67

7.3 Sensitivity to skewness and kurtosis

Considering the NIG distribution of the asset returns and portfolio returns has a strong influence on the optimal choice of the saver in comparison to considering the normal distribution. Here we analyze the impact of the second two moments of the assets on optimal choice and the final wealth properties. We fix the properties of the bond, such that we can better observe a change caused by just one model parameter. We consider the bonds characteristic - mean, volatility, skewness and kurtosis and stocks characteristics - mean and volatility to be fixed while the stock's skewness and stock's kurtosis will change. We thus analyze the impact of the stock's skewness - negative to positive - for different levels of stock's kurtosis on the optimal choice of the portfolio and properties of the final wealth.

The left side Figure 7.5 illustrates the optimal choice evolution for different values of stock's skewness, $\mathcal{S}(S) = -0.7, 0, 0.7$ for $\mathcal{K}(S) = 6$. The lower value of skewness leads to more conservative decisions which directly relates to lower expected value of the wealth. We can observe that the time point when the saver starts to add the bonds to portfolio is the same for all skewness values. The same behavior but for dependence of stock's kurtosis is depicted on the right hand side. The skewness is set to $\mathcal{S}(S) = -0.2$ and kurtosis sequentially to $\mathcal{K}(S) = 4, 6, 8$. The higher is the kurtosis the more conservative portfolios are preferred.

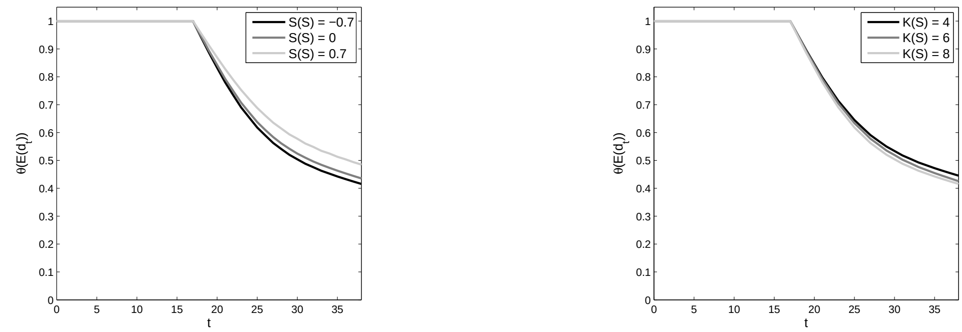


Figure 7.3 Evolution of optimal choice θ for different levels of stock's skewness (on the left) with $\mathcal{K}(S) = 6$ and for different levels of stock's kurtosis (on the right) with $\mathcal{S}(S) = -0.2$).

Higher skewness leads to higher expected wealth while the kurtosis lowers it. The 10% quantile increases with skewness and decreases with kurtosis. The same hold for the 90% quantile.

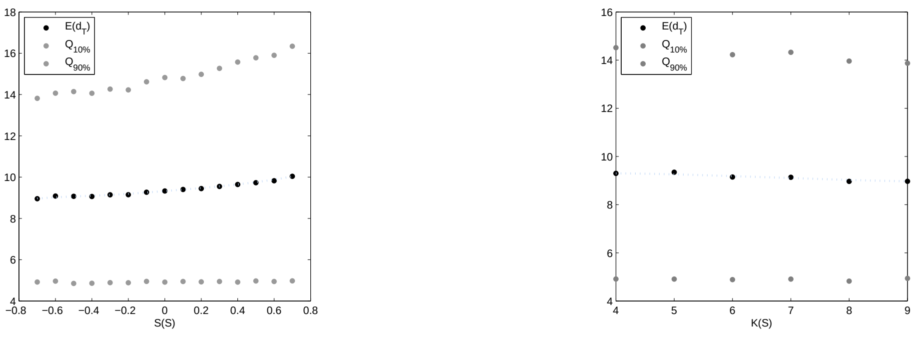
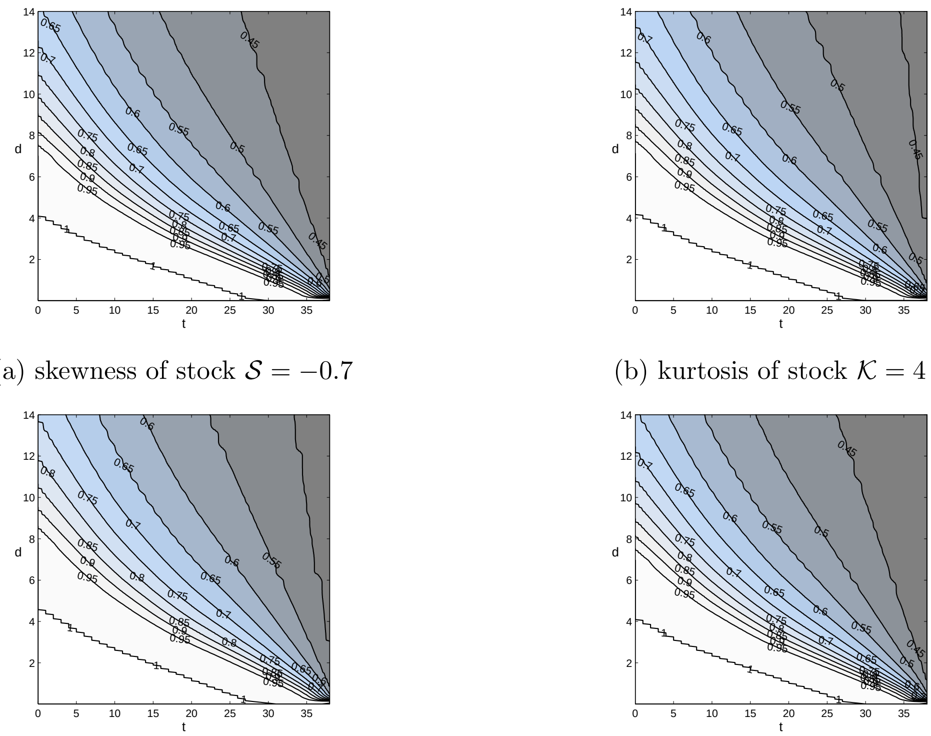


Figure 7.4 Evolution of the expected wealth at the final time horizon (with 10% and 90% quantile) in dependence of the stock's skewness (on the left) and kurtosis (on the right).

The numerical solution of the optimal decision problem depicted on the contour graph (dependence of θ on time t and level of savings d) shows that the skewness move the contours clockwise, Figure 7.5 on the left side, while for the kurtosis the contours move counterclockwise, on the right. It indicates that higher skewness supports higher proportion of stocks in portfolio and on the contrary higher kurtosis supports higher proportion of bonds.



(a) skewness of stock $\mathcal{S} = -0.7$

(b) kurtosis of stock $\mathcal{K} = 4$

(c) skewness of stock $\mathcal{S} = 0.7$

(d) kurtosis of stock $\mathcal{K} = 8$

Figure 7.5 Evolution of the optimal θ in dependence of the stock's skewness and kurtosis.

8 Conclusion

We have presented an investment model with regular contributions for determining the optimal investment opportunity. We have analyzed the model for needs of the risk management in pension system formulated as a dynamic stochastic accumulation model for determining the optimal value of the stock to bond proportion in the pension saving decision. We aimed to introduce the fat tailed distributions for asset returns' modeling and analyze the impact on the saver's preferences.

We have focused on the normal inverse Gaussian distribution among the generalized hyperbolic distributions and studied the S&P500 Index in more detail and similarly to findings of Onalan in [8], we have concluded that the log returns do not follow the normal distribution. They are characterized by higher kurtosis than is typical for the normal distribution and are skewed to the left. The NIG distribution has four parameters which can be used to express first four moments. We have showed how the parameters as well as the moments affect the density shape.

The biggest drawback of NIG distribution follows directly from its convolution property defined only for independent variables exhibiting same shape parameters which is a very restrictive condition. We therefore study the behavior of generating the dependent NIG random variables using the approach usually applied for generating the dependent normally distributed variables, i.e by means of Choleski decomposition. We have analyzed the properties of replicated time series defined by their moments and based on our observations we have introduced the algorithm for generating such mixtures. The crucial parameters showed to be the correlation and the kurtosis.

The NIG distribution exhibits a specific shape and requires also a special treatment by numerical approximation of the formulated Belmann problem. The numerical procedure has to be specified with a focus on needs of NIG distribution or generally distribution with fat tails and high kurtosis. The fat tail requires bigger domain for the space grid which might be computationally expensive. The need of the proper boundary condition is therefore very strong. We derived a heuristic of a boundary condition specific for our type of problem. We have also offered a special interpolation inside the grid based on the shape of the utility function allowing the use of a less dense grid.

The results of our algorithm for the formulated problem suitable for distributions with fat tails were provided with the numerical parameters adopted from [4] in order to demonstrate the algorithm's correctness. We calculated the next two moments of the assets representing the stocks and bonds in portfolio and showed the impact of adding the skewness and the kurtosis on the optimal choice of the saver during the saving time period.

Considering the NIG distribution for the asset returns, the saver tends to keep the maximal stock-to-bond proportion much longer than by assuming normal distribution, however the choice in the last decision year is comparable. This causes that the expected wealth grows faster in first years and in years, when the portfolio is more conservative, the growth does not have to be so high to higher the absolute value of the savings. The expected wealth by considering the NIG distribution is therefore higher than by normal distribution.

The regulation introduced by the Government strongly influences the choice of the saver who would by this parameter setting decide for more risky portfolio. The expected final wealth is lower for regulated investments but lead to lower volatility. In our study we have focus mainly on the distribution of the final accumulated wealth which is characterized by positive skewness since the yearly contribution shifts the savings always in a positive sense. From risk point of view it is more interesting to analyze the left tail of the distribution, the value-at-risk measure that the volatility. From the distribution shape for regulated and non-regulated case and calculated quantiles we can observe that the regulation does not lower the risk. However the difference between the expected value and VaR is higher for non-regulated case. Considering this risk measure the regulation can be understood as a defense from saver's disappointment of not succeeding the expectation and especially of achieving low value compared to expectation. The more regulations influence the choice of the saver the higher is the difference in peak settlement and quantiles of the final wealth distributions.

The same qualitative behavior can be observed by both examined distributions and also for different risk aversion coefficients. As expected, with increasing risk aversion the investor tends to take more conservative decisions for same time and level of accumulated sum. More conservative portfolio leads to lower expected final wealth and lower volatility.

The aim of the thesis was to study the impact of the skewness and the kurtosis of the portfolio assets on the optimal choice of the investor and on his expectation. In our sensitivity analysis we consider again the same numerical and model parameters. We fix the properties of one asset and change the skewness and kurtosis of the second asset in order to better observe a change caused by just one model parameter. We consider the bonds characteristic - mean, volatility, skewness and kurtosis and stocks characteristics - mean and volatility to be fixed while the stock's skewness and stock's kurtosis change. We have showed that the lower value of skewness leads to more conservative decisions leading to lower expected value of the wealth. The time point when the saver starts to add the bonds to portfolio keeps the same for all skewness values. On the other hand lower kurtosis asks for more risky portfolio.

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Co-solver in grants

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