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Analytical and Numerical Approximative Methods for solving Multifactor Models for pricing of Financial Derivatives

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Contents

1 Abstracts

English

The thesis covers different approaches used in current modern computational finance. Analytical and numerical approximative methods are studied and discussed. Effective algorithms for solving multi-factor models for pricing of financial derivatives have been developed.

The first part of the thesis is dealing with modeling of aspects and focuses on analytical approximations in short rate models for bond pricing. We deal with a two-factor convergence model with non-constant volatility which is given by two stochastic differential equations (SDEs). Convergence model describes the evolution of interest rate in connection with the adoption of the Euro currency. From the SDE it is possible to derive the PDE for bond price. The solution of the PDE for bond price is known in closed form only in special cases, e.g. Vasicek or CIR model with zero correlation. In other cases we derived the approximation of the solution based on the idea of substitution of constant volatilities, in solution of Vasicek, by non-constant volatilities. To improve the quality in fitting exact yield curves by their estimates, we proposed a few changes in models. The first one is based on estimating the short rate from the term structures in the Vasicek model. We consider the short rate in the European model for unobservable variable and we estimate it together with other model parameters. The second way to improve a model is to define the European short rate as a sum of two unobservable factors. In this way, we obtain a three-factor convergence model. We derived the accuracy for these approximations, proposed calibration algorithms and we tested them on simulated and real market data, as well.

The second part of the thesis focuses on the numerical methods. Firstly we study Fichera theory which describes proper treatment of defining the boundary condition. It is useful for partial differential equation which degenerates on the boundary. The derivation of the Fichera function for short rate models is presented. The core of this part is based on Alternating direction explicit methods (ADE) which belong to not well studied finite difference methods from 60s years of the 20th century. There is not a lot of literature regarding this topic. We provide numerical analysis, studying stability and consistency for convection-diffusion-reactions equations in the one-dimensional case. We implement ADE methods for two-dimensional call option and three-dimensional spread option model. Extensions for higher dimensional Black-Scholes models are suggested. We end up this part of the thesis with an alternative numerical approach called Trefftz methods which belong to Flexible Local Approximation MEthods (FLAME). We briefly outline the usage in computational finance.

Slovak

Práca popisuje rôzne prístupy používané v súčasnom modernom oceňovaní finančných derivátov. Zaoberáme sa analytickými a numerickými aproximačnými metódami. Vyvinuli sme efektívne algoritmy riešenia viacfaktorových modelov oceňovania finančných derivátov.

Prvá časť práce sa zaoberá modelovaním rôznych aspektov a je zameraná na analytické aproximácie cien dlhopisov v modeloch krátkodobých úrokových mier. Zaoberáme sa dvojfaktorovým konvergenčným modelom s nekonštantnom volatilitou, ktorý je daný dvomi stochastickými diferenciálnymi rovnicami. Konvergenčný model opisuje vývoj úrokovej miery v súvislosti s prijatím eura. Zo stochastickej diferenciálnej rovnice je možné odvodiť parciálnu diferenciálnu rovnicu pre cenu dlhopisu. Riešenie parciálnej diferenciálnej rovnice pre cenu dlhopisu v uzavretej forme je známe iba v špeciálnych prípadoch, napr. Vašíčkov model alebo CIR model s nulovou koreláciou. V ostatných prípadoch, sme odvodili aproximáciu riešenia založenú na myšlienke substitúcie konštantných volatilít, v riešení Vašíčkovho modelu, nekonštantnými volatilitami. Z dôvodu zlepšenia kvality zhody odhadnutých a presných výnosových kriviek sme navrhli niekoľko zmien v modeloch. Prvá z nich je založená na odhade výnosových kriviek z časovej štruktúry úrokových mier vo Vašíčkovom modeli. Krátkodobú úrokovú mieru považujeme za nepozorovateľnú premennú a odhadujeme ju spolu s ostatnými parametrami modelu. Druhý spôsob ako vylepšiť model je

definovanie európskej krátkodobej úrokovej miery ako súčtu dvoch nepozorovateľných faktorov. Týmto spôsobom získavame trojfaktorový konvergenčný model. Odvodili sme presnosť aproximácie, navrhli sme postup kalibrácie a testovali sme ho na simulovaných a reálnych trhových dátach.

Druhá časť práce sa zameriava na numerické metódy. Najskôr študujeme Ficherovu teóriu, ktorá popisuje správne zaobchádzanie a definovanie okrajových podmienok pre parciálne diferenciálne rovnice, ktoré degenerujú na hranici. V práci uvádzame odvodenie Ficherových podmienok pre modely krátkodobých úrokových mier. Jadrom tejto časti sú ADE (alternating direction explicit) metódy zo 60. rokov 20. storočia, ku ktorým sa nenachádza veľa literatúry. V práci je obsiahnutá numerická analýza, štúdium stability a konzistencie pre konvekčno-difúzno-reakčnú rovnicu v jednorozmernom prípade. ADE metódy implementujeme pre dvojrozmerné call opcie a trojrozmerné spread opcie. Navrhujeme rozšírenia na viacrozmerné prípady Black-Scholesovho modelu. Túto časť práce ukončujeme alternatívnou metódou nazývanou Trefftz, ktorá patrí medzi Flexible Local Approximation MEthods (FLAME).

German

Die Doktorarbeit beinhaltet verschiedene Methoden, die in der heutigen modernen Finanzmathematik eingesetzt werden. Es werden analytische und numerische Approximationsmethoden analysiert und diskutiert, sowie effektive Algorithmen für Multifaktormodelle zur Bewertung von Finanzderivaten entwickelt.

Der erste Teil der Doktorarbeit behandelt Modellierungsaspekte und ist auf die analytische Approximation von Zinssatzmodellen im Anleihenmarkt fokussiert. Wir behandeln ein Zweifaktorkonvergenzmodell mit nichtkonstanter Volatilität, das durch zwei stochastische Differentialgleichungen (SDG) gegeben ist. Das Modell beschreibt die Entwicklung von Zinsraten in Verbindung mit dem Eurowechselkurs. Ausgehend von der SDG ist es möglich eine partielle Differentialgleichung (PDG) für den Anleihekurs herzuleiten. Eine Angabe der Lösung der PDG ist nur in Einzelfällen in geschlossener Form möglich, z.B. im Vasicek or CIR Modell mit Korrelation null. In anderen Fällen haben wir eine Approximation an die Lösung des CIR Modells durch Ersetzen der konstanten Volatilität duch eine flexible Volatilität erhalten. Um eine höhere Genauigkeit bei der Anpassung an die reale Zinskurve zu erhalten, haben wir einige Änderungen innerhalb des Modells vorgeschlagen. Die erste basiert dabei auf der Schätzung des Momentanzinses durch die Zinsstrukturkurse innerhalb des Vasicek-Modells. Wir betrachten den Momentanzins im europäischen Modell für eine unbeobachtbare Variable und schätzen diese zusammen mit den anderen Modellparametern. Als zweite Verbesserungsmöglichkeit des Modells betrachten wir den europäischen Momentanzins als Summe von zwei unbeobachtbaren Prozessen. Auf diesem Wege erhalten wir ein Dreifaktorkonvergenzmodell. Wir zeigen die Genauigkeit dieser Approximationen, schlagen Kalibirierungsalgorithmen vor und testen die Modelle an simulierten, sowie realen Marktdaten.

Der zweite Teil der vorliegenden Arbeit beschäftigt sich mit numerischen Methoden. Zuerst erläutern wir die Fichera-Theorie, die eine systematische Untersuchung von Randbedingungen erlaubt. Sie ist bei partiellen Differentialgleichungen, die am Rand degenerieren, von großem Nutzen. Es wird die Fichera-Funktion für Zinssatzmodelle hergeleitet. Den Kern der Doktorarbeit bilden Alternating Direction Explicit (ADE) Verfahren, aus den 60er des 20. Jahrhunderts die zu den nicht ausgiebig untersuchten Verfahren zählen. Daher existiert heute nur sehr wenig Literatur zu diesem Thema. Wir führen eine numerische Analyse durch und untersuchen die Stabilitäts- und Konsistenzeigenschaften für Konvektions-Diffusions-Reaktions Gleichungen in einer Raumdimension. Wir implementieren ADE Methoden für zweidimensionale Call-Optionen und dreidimensionale Spreadoptionsmodelle. Zusätzlich werden Erweiterungen für das höherdimensionale Black-Scholes-Modell vorgeschlagen. Wir beenden diesen Abschnitt der Doktorarbeit mit einer alternativen numerischen Methode, der sogenannten Trefftz-Methode, die zu der Klasse der Flexible Local Approximation MEthods (FLAME) gehört. Wir erläutern kurz ihre Nutzung im Rahmen der Finanzmathematik.

2 Aims of the dissertation thesis

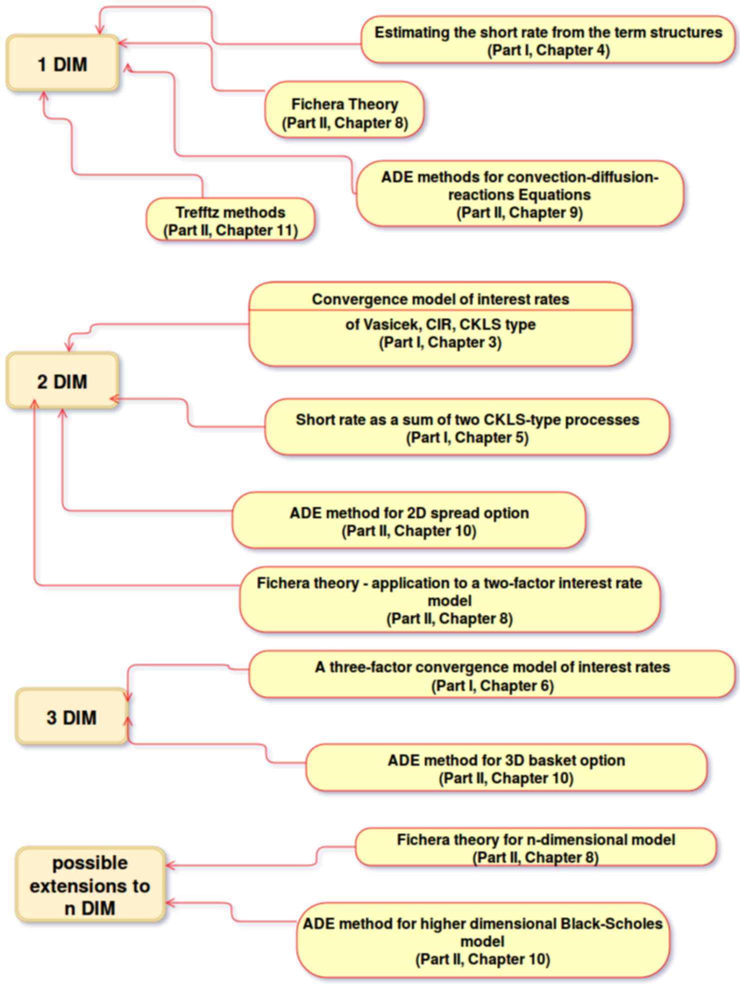
The goal of the thesis is to provide a wide scope of techniques used in computational finance. On the one hand we see importance of the analytical techniques, on the other hand we tackle with numerical schemes. Another goal is to provide models and their solutions which are easy implementable. The better model, the better description of the reality. But the more complex model, the more troubles. Extension to higher dimensional (or nonlinear models) is necessary but or goal is to keep in mind the simpler model, the better. We do not want to deal with calibration and solving too complex models, because something it is even not possible. The suggested model or scheme must be tractable. In recent years we are witnesses of the negative interest rates in the whole European union. This fact must be considered and included to the all the models created in the way that they are capable to distinguish and cover all the situations. If the case is too complicated, we should provide an implementation in the way where it is possible and easy to provide parallelization of the algorithm. Computational finance is applied science but it requires knowledge from various fields in mathematics: SDEs, PDEs, analytical techniques, numerical analysis, optimization, programming and everything with having some knowledges from pricing of financial derivatives, such as options and bonds. Goal of this thesis is to cover all these subjects and suggest the effective methods for the given task.

To summarize the aim of the dissertation thesis we say we deal with the mathematical analysis of the multifactor models for pricing of financial derivatives which includes:

- short rate models for bond pricing
  - convergence model of an interest rate
  - choice of a suitable model for the European and the domestic interest rate
  - a two-factor short rate model and its extension to a three-factor model
  - suggestion of the analytical approximation
  - bond pricing
  - implementation of a simulation analysis
  - calibration of these models
- numerical solutions of the second order parabolic PDE
  - Fichera theory (boundary conditions)
  - finite difference method, or some other alternative approaches
  - Alternating Direction Explicit (ADE) method
  - numerical analysis of ADE schemes incluuding stability and consistency analysis
- considering one-dimensional and higher dimensional models
  - suggestion of an effective algorithms for dealing with these models

3 Structure of the thesis and included techniques

In the thesis we deal with one-dimensional (or in other terminology one-factor), two-dimensional and three-dimensional models and we outline extensions to higher dimensional cases. Sorting according to dimensionality is displayed in the following diagram which models and analytical and numerical approximative methods are included in the thesis.



The chapters are mostly based on the published results and each chapter is accesible in any order allowing a swift reading to readers. For reader interested in numerical analysis we refer to the second part of the thesis, for reader interested more in analytical techniques we recommend to read the first part of the thesis. The thesis does not have extended theoretical part but it is a collection of own research results.

In the following text we explain which kind of models are studied in the thesis. Dynamics of the stock price, interest rate, volatility are described by SDEs which can be solved using analytical methods or numerical simulations. From SDE we can derive the corresponding partial differential equation (PDE) which describes the price of the bond, or option. In our work we deal with the analytical and numerical PDE approaches. Interest rate modeling using short rate models, analytical approximations for bond pricing and its accuracy are discussed in detail in the first part of the thesis. Second part of the thesis is focused on the efficient numerical solutions of higher dimensional option pricing problems which are described by a parabolic PDE, also called also Black-Scholes (BS) model.

Models which are studied in this work, are described by the SDE:

$$\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W,$$

where W is Wiener process. Function $\mu(X_t,t)$ is the trend or drift of the equation and $\sigma(X_t,t)$ describes fluctuations around the drift. A solution of this SDE is a stochastic process X_t. For scalar X_t we have one-factor models, for vector valued X_t we deal with multi-factor models. Dynamics of the evolution of the process X_t is described by SDE, where X_t can represent an underlying asset, usually a stock price, but it can also be interest rate or volatility. In case X_t is a short rate, derived PDE represents equation for pricing of bonds. In case X_t is a stock price, derived PDE is an equation for option pricing, called also Black-Scholes model. If additionally there is given a stochastic volatility, we get the Heston model; and if there is given also a stochastic interest rate, it leads to the Heston-Hull-White model.

4 Modeling of the interest rate in short rate models, bond pricing and its analytical approximations

A discount bond is a security which pays its holder a unit amount of money at specified time T (called maturity). $P(t,T)$ is the price of a discount bond with maturity T at time t. It defines the corresponding interest rate $R(t,T)$ by the formula

$$P(t,T) = e^{-R(t,T)(T-t)}, \text{i.e. } R(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

A zero-coupon yield curve, also called term structure of interest rates, is then formed by interest rates with different maturities. Short rate (or instantaneous interest rate) is the interest rate for infinitesimally short time. It can be seen as the beginning of the yield curve: $r(t) = \lim_{t\to T^-} R(t,T)$. For a more detailed introduction to short rate modeling see e.g. [2], [15].

Short rate models are formulated by stochastic differential equation (SDE) for a variable X:

$$\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W$$

which defines the short rate $r = r(X_t)$. Here W is a Wiener process, function $\mu(X_t,t)$ is the trend or drift part and the volatility $\sigma(X_t,t)$ represents fluctuations around the drift. Choosing different drift $\mu(X_t,t)$ and volatility $\sigma(X_t,t)$ leads to various one-factor models (where X_t is a scalar) and multi-factor models (where X_t is a vector). In short rate models, bond prices and term structures of interest rates are determined by the parameters of the model and the current level of the instantaneous interest rate (so called short rate).

4.1 Convergence model of CKLS type

A convergence model of interest rates explains the evolution of the domestic short rate in connection with the European rate. The first model of this kind was proposed by Corzo and Schwartz in 2000 and its generalizations were studied later. Its dynamics is described by two stochastic differential equations - the domestic and the European short rate. We consider a model in which the risk-neutral drift of the European short rate r_e is a linear function of r_e, risk-neutral drift of the domestic short rate r_d is a linear function of r_d and r_e and volatilities take the form $\sigma_e r_e^{\gamma_e}$ and $\sigma_d r_d^{\gamma_d}$, i.e.

$$\begin{aligned} \mathrm{d}r_d &= (a_1 + a_2 r_d + a_3 r_e)\mathrm{d}t + \sigma_d r_d^{\gamma_d} dW_d, \\ \mathrm{d}r_e &= (b_1 + b_2 r_e)\mathrm{d}t + \sigma_e r_e^{\gamma_e} dW_e, \end{aligned} \tag{1}$$

where $Cov[dW_d, dW_e] = \rho dt$. Parameters $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}, \sigma_d, \sigma_e > 0, \gamma_d, \gamma_e \geq 0$ are given constants and $\rho \in (-1,1)$ is a constant correlation between the increments of Wiener processes dW_d and dW_e. We will refer to this model as *two-factor convergence model of Chan-Karolyi-Longstaff-Sanders (CKLS) type*. The domestic bond price $P(r_d, r_e, \tau)$ with the maturity τ satisfies the PDE:

$$\begin{aligned} &-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} \\ &+ \frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\frac{\partial^2 P}{\partial r_d^2} + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\frac{\partial^2 P}{\partial r_e^2} + \rho \sigma_d r_d^{\gamma_d} \sigma_e r_e^{\gamma_e} \frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0, \end{aligned} \tag{2}$$

for $r_d, r_e > 0, \tau \in (0,T)$, with the initial condition $P(r_d, r_e, 0) = 1$ for $r_d, r_e > 0$. Unlike for Vasicek and uncorrelated CIR model, in this case it is not possible to find solution in the separable form. For this reason, we are seeking for an approximative solution (2).

In our paper paper [7] substituting its constant volatilities by instanteneous volatilities we obtain an approximation of the solution for a general model. We compute the order of accuracy for this approximation, propose an algorithm for calibration of the model and we test it on the simulated and real market data. On the one hand, suggested approximation and calibration algorithm provide reasonably accurate results, which was proved, but on the other hand, by comparing accuracy of estimated and exact yield curves we did not achieve wanted results. Improvements are suggested in the following subsections.

4.2 Estimating the short rate from the term structures

As an improvement for modeling of the interest rate we suggest the idea of Estimating the short rate from the term structures in the Vasicek model in our paper [13]. The idea of this approach is to use observable market term structures to calibrate the model. We use the weighted mean square error where a difference between observed yields and computed yields, using Vasicek model, is minimized. The calibration procedure was proposed and the application on simulated and real market data was performed. We tried to find out the relation between the estimated short rate and market overnight. It is interesting to deal with if it is really necessary to consider the short rate for the unobservable variable and estimate it together with other model parameters. We find out that the estimated short rate is robust when the set of maturities of the interest rate is being changed.

4.3 Short rate as a sum of CKLS-type processes

As an another improvement for modeling of the European interest rate we suggest the short rate model of interest rates, in which the short rate is defined as a sum of two stochastic factors discussed in our paper [5]. Each of these factors is modeled by a stochastic differential equation with a linear drift and the volatility proportional to a power of the factor. We show calibration methods which - under the assumption of constant volatilities – allow us to estimate the term structure of interest rate as well as the unobserved short rate, although we are not able to recover all the parameters. We apply it to real data and show that it can provide a better fit compared to a one-factor model. A simple simulated example suggests that the method can be also applied to estimate the short rate even if the volatilities have a general form. Therefore we propose an analytical approximation formula for bond prices in such a model and derive the order of its accuracy.

4.4 A three-factor convergence model of interest rates

Combining two approaches from [7] and [5] we suggested a three-factor convergence model of interest rates [21]. In all the previous models, the European rates are modeled by a one-factor model. This, however, does not provide a satisfactory fit to the market data. A better fit can be obtained using the model, where the short rate is a sum of two unobservable factors. We model European rate by 2 SDEs and the domestic interest rate by 1 stochastic differential equation. Therefore, we build the convergence model for the domestic rates based on this evolution of the European market. We propose the following model for the joint dynamics of the European r_e and domestic r_d instantaneous interest rate. The European rate $r_e = r_1 + r_2$ is modeled as the sum of the two mean-reverting factors r_1 and r_2, while the domestic rate r_d reverts to the European rate. Volatilities of the processes are assumed to have a general CKLS form. Hence

$$
\begin{aligned}
dr_1 &= \kappa_1(\theta_1 - r_1)dt + \sigma_1 r_1^{\gamma_1} dw_1 \\
dr_2 &= \kappa_2(\theta_2 - r_2)dt + \sigma_2 r_2^{\gamma_2} dw_2 \\
dr_d &= \kappa_d((r_1 + r_2) - r_d)dt + \sigma_d r_d^{\gamma_d} dw_d
\end{aligned}
$$

with $Cor(dw) = \mathscr{R}dt$, where $dw = (dw_1, dw_2, dw_d)^T$ is a vector of Wiener processes with correlation matrix $\mathscr{R}$, whose elements (i.e., correlations between r_i and r_j) we denote by ρ_{ij}. Figure 1 and Figure 2 show the evolution of the factors and the interest rates for the following set of parameters: $\kappa_1 = 3, \theta_1 = 0.02, \sigma_1 = 0.05, \gamma_1 = 0.5, \kappa_2 = 10, \theta_2 = 0.01; \sigma_2 = 0.05, \gamma_2 = 0.5, \kappa_d = 1, \sigma_d = 0.02, \gamma_d = 0.5, \rho_{ij} = 0$ for all i, j.

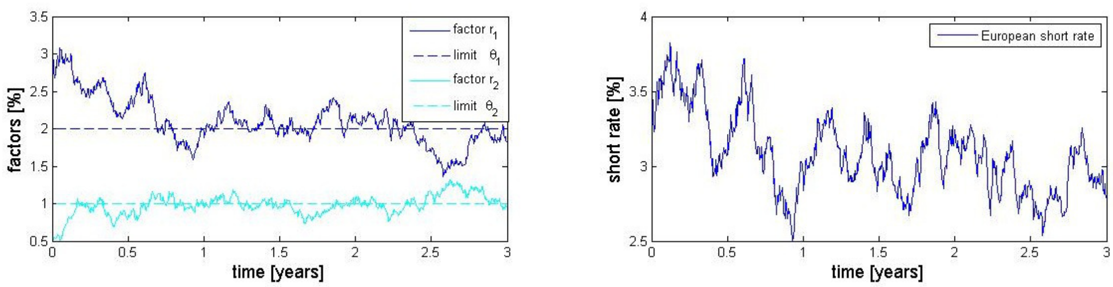


Figure 1: Simulation of the factors r_1, r_2 (left) and the European short rate $r_e = r_1 + r_2$ (right).

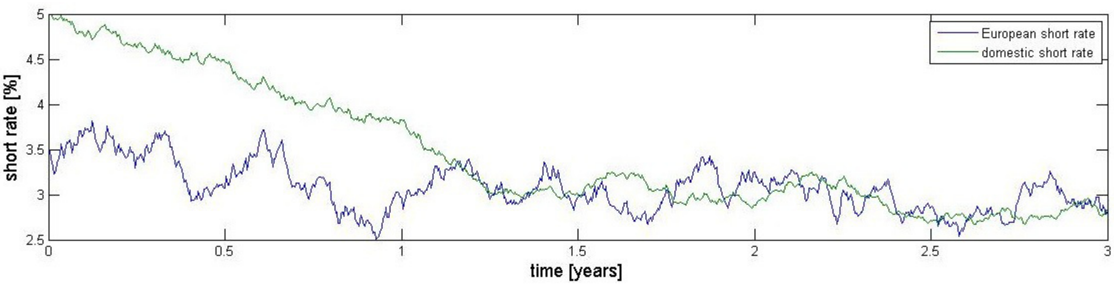


Figure 2: Simulation of the European short rate r_e and the domestic short rate r_d.

We study the prices of the domestic bonds in this model which are given by the solution of the PDEs. In general, it does not have an explicit solution. Hence we suggest an analytical approximative formula and derive the order of its accuracy in a particular case.

5 Numerical methods

We focus on the *Alternating Direction Explicit (ADE) methods*, as an efficient scheme, which can be used for a wide range of financial problems. Designing the numerical scheme, we do not only need to take care for the choice of a mesh, but have to choose the boundary conditions carefully, as well. Because of the issue with the boundary conditions we have studied the *Fichera theory*, which helped us to distinguish how to define boundary conditions for equations degenerating on the boundary. According to the *sign of the Fichera function*, we chose which kind of boundary conditions need to be supplied. The second issue about boundary condition is the influence of the stability of the numerical scheme. Since the matrix approach also includes boundary conditions, we prefer to use it for the stability analysis instead of the von Neumann stability analysis.

5.1 Results of application of Fichera theory in financial applications

Choosing set of parameters $\kappa = 0.5$, $\theta = 0.05$, $\sigma = 0.1$, $\gamma = 0.5$ in CIR model $\mathrm{d}r = \kappa(\theta - r)\,\mathrm{d}t + \sigma r^{\gamma}\mathrm{d}W_t$, we get at $r = 0$ a positive Fichera function $b = \kappa\theta - \sigma^2/2 = 0.02 > 0$. This is equivalent with the statement that the Feller condition is satisfied. According to the Fichera theory, as soon as it is outflow part of boundary, we must not supply BCs. In this example in Fig. 3 and Fig. 5 and Table 1, we intentionally supplied BCs in an 'outflow' situation when we should not in order to illustrate what might happen if one disregards the Fichera theory. In the evolution of the solution we can observe a peak and oscillations close to the boundary. In Fig. 5 we plot the relative error, which is reported also in the Table 1.

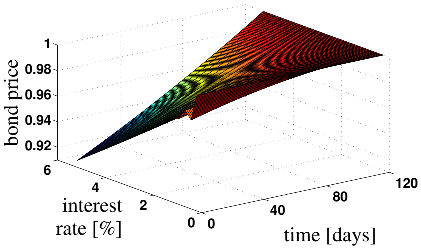


Figure 3: Numerical solution, Dirichlet BC

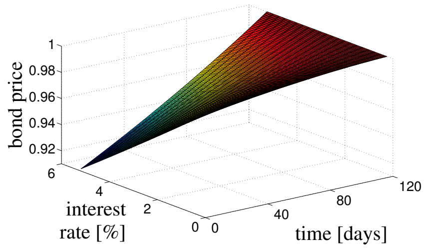


Figure 4: Numerical solution, without BC

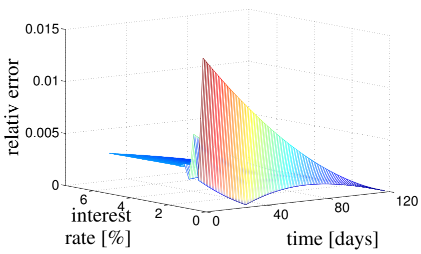


Figure 5: Relative error, case with Dirichlet BC

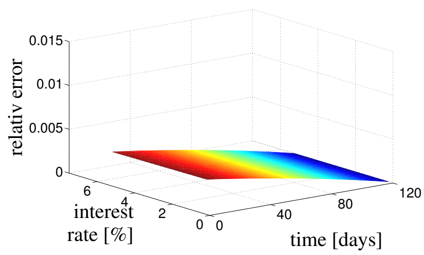


Figure 6: Relative error, case without BC

In our example we used the same parameters, but with or without defining Dirichlet BC. Here, "without BC" means that we used for the numerical BC the limit of the interior PDE for $r \to 0$. The corresponding results are shown on the right hand side, in Fig. 4, Fig. 6 and the relative errors are recorded in Table 1. For the numerical solution we used the implicit finite difference method from [12]. The reference solution is obtained either as the analytic solution for the CIR model

Table 1: Relative error, case with BC

time[days]	relative error
1	0.0147
40	0.0079
80	0.0029
120 (maturity)	0

Table 2: Relative error, case without BC

time[days]	relative error
1	0.0039
40	0.0029
80	0.0015
120 (maturity)	0

($\gamma = 0.5$, if Feller condition is satisfied), c.f. [2] or in all other cases using a very fine resolution (and suitable BCs). The conditions at outflow boundaries are obtained by studying the limiting behavior of the interior PDE or simply by horizontal extrapolation of appropriate order. Recall that negative values of the Fichera function (i.e. an inflow boundary) correspond to not satisfied Feller condition and may destroy the uniqueness of solutions to the PDE. Details about deriving Fichera function for one, two factor short rate models can be found in our proceedings paper [4].

5.2 Explanation of Alternating direction explicit methods

ADE schemes were proposed by Saul'ev [20] in 1957, later developed by Larkin [16], Bakarat and Clark [1] in 1964-66. More recently, these schemes have received some attention by Duffy [11], [10] 2013 and Leung and Osher [17] 2005 who have studied and applied these schemes in both financial modeling and other applications. Numerical analysis of ADE schemes for one dimensional convection-diffusion-reaction equation can be found in our paper [3] and in [17]. Some advantages of the ADE methods are that they can be implemented in a parallel framework and are very fast due to their explicitness; for a complete survey on the advantages and the motivation to use them in a wide range of problems we refer the reader to [9], [10].

We have considered ADE method, that strongly uses boundary data in the solution algorithm and hence is very sensible to incorrect treatment of boundary conditions. We have implemented ADE scheme for solving Black-Scholes equation. By treating the nonlinearity explicitly, it can be applied for nonlinear BS equations, as well. ADE scheme consists from two steps (sweeps). In the first step upward sweeping is used and in the second step downward sweeping is used and there are combined after each time step. To our knowledge, the ADE scheme has not been applied to nonlinear PDEs before. It can compete to the Crank-Nicolson scheme, Alternating Direction Implicit (ADI) and LOD splitting method. The sweeping procedure is done from one boundary to another and vice versa.

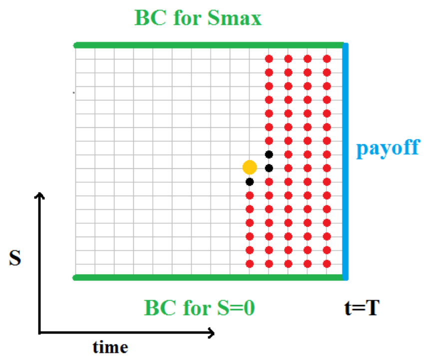


Figure 7: Upward sweep

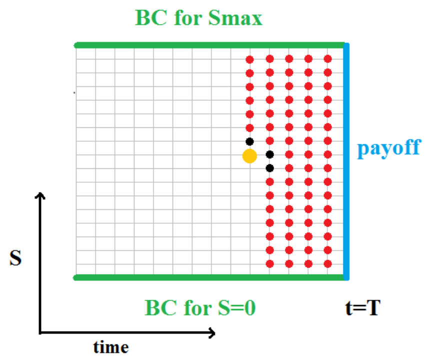


Figure 8: Downward sweep

Figures 7 and 8 show a grid for calculating the price of a call option in the BS model. The blue line represents payoff as an initial condition and the green lines are given by the Dirichlet boundary conditions for small and big asset values. The calculation is made by stepping backward in time. To calculate the value of the yellow point in the upward sweep, we use the black values. We can see that we do not use only values from the previous time level but also already known values from the current time level, what preserves the explicitness of this sweep. Next, for the same time level, we repeat the process with the downward sweep. To complete the current time level, we combine the intermediate solutions from the upward and downward sweep by averaging, which gives the final approximation at this time level. The schematic algorithm we can express in the following way:
For $\mathbf{n = 0, 1, \ldots, N-1}$

1. Init: $u_j^n = c_j^n, \quad d_j^n = c_j^n, \qquad j = 1, ..., J-1$
2. Upward solution, $u_j^{n+1} \qquad j = 1, \ldots, J-1$
3. Downward solution, $d_j^{n+1} \qquad j = J-1, \ldots, 1$
4. Combination $\quad c^{n+1} = \frac{u^{n+1}+d^{n+1}}{2}$

For the discretization of the *diffusion term* we use, c.f. [20]

$$\begin{aligned} \frac{\partial^2 v(x_j,t_n)}{\partial x^2} &\approx \frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{h^2}, \qquad & j = 1, \ldots, J-1 \\ \frac{\partial^2 v(x_j,t_n)}{\partial x^2} &\approx \frac{d_{j+1}^{n+1} - d_j^{n+1} - d_j^n + d_{j-1}^n}{h^2}, \qquad & j = J-1, \ldots, 1. \end{aligned} \tag{3}$$

In order to obtain a symmetric scheme we use the following approximations of the *reaction term*, the same for the upward and downward sweep

$$v(x_j,t_n) \approx \frac{u_j^{n+1} + u_j^n}{2}, \quad j = 1, \ldots, J-1; \quad \text{and} \quad v(x_j,t_n) \approx \frac{d_j^{n+1} + d_j^n}{2}, \quad j = J-1, \ldots, 1. \tag{4}$$

Different approximations of the *convection term* are possible [17], [8]. In the following we state three of them. First, Towler and Yang [22] used special kind of centered differences

$$\frac{\partial v(x_j,t_n)}{\partial x} \approx \frac{u_{j+1}^n - u_{j-1}^{n+1}}{2h}, \quad j = 1, \ldots, J-1; \quad \text{and} \quad \frac{\partial v(x_j,t_n)}{\partial x} \approx \frac{d_{j+1}^{n+1} - d_{j-1}^n}{2h}, \quad j = J-1, \ldots, 1. \tag{5}$$

More accurate approximations were proposed by Roberts and Weiss [19], Piacsek and Williams [18]

$$\begin{aligned} \frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{u_{j+1}^n - u_j^n + u_j^{n+1} - u_{j-1}^{n+1}}{2h}, \qquad & j = 1, \ldots, J-1, \\ \frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{d_{j+1}^{n+1} - d_j^{n+1} + d_j^n - d_{j-1}^n}{2h}, \qquad & j = J-1, \ldots, 1. \end{aligned} \tag{6}$$

As a third option we will use *upwind* approximations combined with the ADE technique. Since we have in mind financial applications we will focus on left going waves, i.e. negative sign before convection term. Right going waves (positive sign before convection term) are treated analogously.

The well-known first order approximation reads

$$\frac{\partial v(x_j,t)}{\partial x} \approx \frac{v_{j+1}(t)-v_j(t)}{h} \qquad j=J-1,\dots,1, \tag{7}$$

and the forward difference of second order [24]

$$\frac{\partial v(x_j,t)}{\partial x} \approx \frac{-v_{j+2}(t)+4v_{j+1}(t)-3v_j(t)}{2h}, \qquad j=J-1,\dots,1. \tag{8}$$

Applying the ADE time splitting we obtain for the upwind strategy (7)

$$\begin{aligned} \frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{u^n_{j+1}-u^n_j}{h}, && j=1,\dots,J-1,\\ \frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{d^n_{j+1}-d^n_j+d^{n+1}_{j+1}-d^{n+1}_j}{2h}, && j=J-1,\dots,1, \end{aligned} \tag{9}$$

and for the second order approximation

$$\begin{aligned} \frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{-u^n_{j+2}+4u^n_{j+1}-3u^n_j}{2h}, && j=1,\dots,J-1,\\ \frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{-d^n_{j+2}+4d^n_{j+1}-3d^n_j-d^{n+1}_{j+2}+4d^{n+1}_{j+1}-3d^{n+1}_j}{4h}, && j=J-1,\dots,1. \end{aligned} \tag{10}$$

We will show that this upwind approximation (9) leads to a stable scheme.

Numerical analysis results focusing on stability and consistency considerations are described in [17] and [3]. In [3] a numerical analysis of convection-diffusion-reaction equation with constant coefficients and smooth initial data is provided. The authors proved that the ADE method applied to the one-dimensional reaction-diffusion equation on a uniform mesh with the discretization of the diffusion according to Saul'ev [20] and the discretization of the convection term following Towler and Yang [22] is unconditionally stable. If a convection term is added to the equation and upwind discretization for this term is used, the ADE scheme is also unconditionally stable c.f. [3].

In the ADE schemes one computes for each time level two different solutions which are referred to as sweeps. Hereby the number of sweeps does not depend on the dimension. It has been shown [3, 11, 17] that for the upward and downward sweep the consistency is of order $O((d\tau)^2+h^2+\frac{d\tau}{h})$ where $d\tau$ is the time step and h denotes the space step. An exceptionality of the ADE method is that the average of upward and downward solutions has consistency of order $O((d\tau)^2+h^2)$. For linear models, unconditional stability results and the $O((d\tau)^2+h^2)$ order of consistency lead to the $O((d\tau)^2+h^2)$ convergence order.

Stability, consistency and convergence analysis can be extended to higher dimensional models.

The straightforward implementation also to nonlinear cases with preserving good stability and consistency properties of the scheme is also a strong advantage. In this paper we show how one can implement this scheme for higher dimensional models by focusing on a linear model. However, one could use this procedure for non-linear models as well. One way how to do it is to solve nonlinear equation in each time level, instead of system of nonlinear equations in case of implicit schemes. Another way is to keep nonlinearity in the explicit form and solve it directly. Powerful tool for nonlinear equations represents also the Alternating segment explicit-implicit and the implicit-explicit parallel difference method [25].

5.3 ADE schemes for multidimensional models

We have suggested and implemented ADE schemes algorithm for two- and three-dimensional models appearing in finance, esp. the multi-dimensional linear BS model. Details are recorded in our paper [6]. ADE schemes have not used before for multidimensional models. We consider two-dimensional (2D) spread options and also three-dimensional (3D) call options. Experimental and theoretical second order of convergence for three dimensional call option model is displayed in the Figure 9.

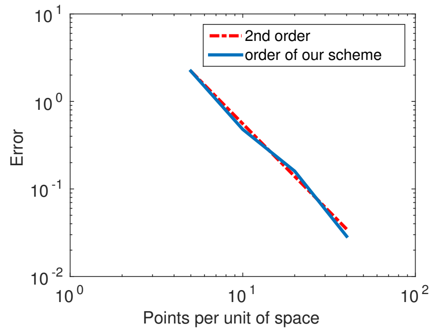


Figure 9: Experimental order of convergence in three dimensional call option model

Approach in our algorithm and its fundamental set-up can be implemented even in higher dimensions. The advantage is that in higher dimensional cases still only 2 sweeps are required. Another advantage of the ADE schemes is that they are suitable method for parallelization.

5.4 Trefftz methods, Flexible local approximation schemes (FLAME)

Goal of this section is to present a short overview on alternative methods for solving the Black-Scholes model. Trefftz methods are represented by Flexible Local Approximation Methods (FLAME). They have been applied in different areas, but not in finance yet. Trefftz schemes are an alternative to traditional methods solving the for Black-Scholes equation. The Trefftz approach may lead to new finite difference schemes. Trefftz functions by definition satisfy the underlying differential equation. Examples for basis functions are exponentials, plane waves, harmonic polynomials, etc. There is a lot of study for stationary problems but how it works for time-dependent problems like the Black-Scholes equation. One example is given in [23] pp.7-8. Here, the time is considered as an additional coordinate. Basis functions are chosen as dependent functions on space and time. Chapter about Trefftz methods serves in the thesis as a proof of concepts that Trefftz methods can be used in different fields. However there is a lot of scope for improvement and suggesting the FLAME scheme with good properties. Big potential of FLAME methods is to generate as good exact solution as possible based on the choice of the basis functions. In a series of experiments we observe some numerical instabilities which are the subject for a deeper study. Suggestions of using Trefftz basis function in another approaches, e.g. Discontinuous Galerkin method are challenging, as well. FLAME has a great deal of flexibility which makes this method competitive. The application to nonlinear equations and usage of nonuniform meshes can be a nice enrichment of these approaches (it can save a lot of computational time, it is convenient to use nonuniform mesh for financial problems; e.g. a mesh according to [14]).

6 Achieved results, conclusion and outlook

Thesis covers both analytical and numerical methods used in computational finance, as it was aimed. It deals with different models, as short rate models for bond pricing or Black-Scholes equations for call and spread option pricing. It covers deep study about proper treatment of boundary conditions.

The first part of the thesis is focused on searching for a suitable approximative solution of the convergence CKLS model. Crucial part of the analytical part of the thesis is based on [7] where accuracy for analytical approximation for two-dimensional CKLS is derived. We suggest an improved approximation with higher accuracy order. A complete analysis of the the approximation, testing on simulated and providing calibration using real market data are included. Since the results with real market were not perfectly satisfactory, we implemented a few improvements. One of them is the estimation of the overnight interest rates based on the modeling from the term structures of the interest rate in Vasicek model in the [13]. Simulations show that the procedure exhibits high precision. When applying it to the real data, we obtain a good fit of the term structures. Another possibility how to improve modeling of the stochastic interest rate model is to suggest the alternative model, where the one-dimensional stochastic process is modeled as a sum of two unobservable processes [5]. Since calibration of the bond yields is dependent on the European data, improvement in fitting of the bond yield in European model will also influence the accuracy of the domestic bond yield curves. We end up this modeling and analytical part of the thesis with three-factor convergence model of interest rate with the [21]. Combining two-factor convergence short rate model and improvement in modeling of one interest rate as a sum of two CKLS-type processes lead to the three-factor model. Calibration algorithms for convergence model of interest rate in deeply described and on real market data provided.

In the second part of the thesis and in the [4] we discuss application of the classical Fichera theory to the resulting degenerated parabolic PDEs from one and two factor short rate models. This theory provides highly relevant information how to supply BCs in these applications. We provide a numerical analysis for ADE methods solving linear convection-diffusion-reaction equations. The stability was investigated by two different approaches. The matrix approach yields unconditional stability in the downward sweep using upwind discretization. The von-Neumann analysis yields unconditional stability of the downward sweep using the Roberts and Weiss approximation. It turned out that the order of consistency is $\mathrm{O}(k^2+h^2+k/h)$ for the upward or downward sweeps, but its combination exhibits an increase order of consistency $\mathrm{O}(k^2+h^2)$. More details can be found in our paper [3]. We suggest the usage of ADE methods to numerically solve higher-dimensional PDEs. We implemented it for the linear 2D and 3D Black-Scholes pricing equation in [6]. The second order of consistency of the implemented ADE method for higher dimensional models was verified experimentally. Also, since the ADE approach is quite suitable to parallelization, an implementation using a parallel computing environment will be envisaged.

In the last Chapter of the thesis we briefly introduce an alternative approach of solving the Black-Scholes equation based on the flexible local approximative schemes, also called Trefftz methods. The results are very preliminary and there is a lot of room for improvements.

The thesis deals with broad scope of numerical and analytical techniques. It brings unique results in form of approximations in closed form formula in short-rate models and brushes up forgotten ADE schemes, brings its numerical analysis and implements it in higher dimensional models. Also some other side results appeared as a surprise which had not been really planned, such as Fichera theory or Trefftz methods, what is of course a positive finding.

7 Own authors publications and its citations

Papers in reviewed journals and proceedings:

1. *Z. Bučková (Zíková), B. Stehlíková, Convergence model of interest rates of CKLS type, Kybernetika 48(3), 2012, 567-586*
   - cited in: N. Ishimura, T. Fujita, M. Nakamura, A model of the instantaneous interest rate in discrete processes, Procedia Economics and Finance 5, 2013, 355–360
2. *J. Halgašová, B. Stehlíková, Z. Bučková (Zíková): Estimating the short rate from the term structures in the Vasicek model, Tatra Mountains Mathematical Publications 61: 87-103, 2014*
3. *Z. Bučková, J. Halgašová, B. Stehlíková: Short rate as a sum of CKLS-type processes, accepted for publication in Proceedings of Numerical analysis and applications conference, Springer Verlag in LNCS, 2016.*
4. *B. Stehlíková, Z. Bučková (Zíková): A three-factor convergence model of interest rates. Proceedings of Algoritmy 2012, pp. 95-104.*
5. *Z. Bučková, M. Ehrhardt, M. Günther: Fichera theory and its application to finance, Proceedings ECMI 2014, Taormina, Sicily, Italy, 2016*
   - cited in: M. C. Calvo-Garrido, M. Ehrhardt, C. Vázquez Cendón: Pricing Swing Options in Electricity Markets with Two Stochastic Factors Using a Partial Differential Equation Approach, Journal of Computational Finance, 2016
6. *Z. Bučková, M. Ehrhardt, M. Günther: Alternating Direction Explicit Methods for Convection Diffusion Equations, Acta Math. Univ. Comenianae, Vol. LXXXI: 309–325, 2015*
7. *Z. Bučková, P. Pólvora, M. Ehrhardt, M. Günther: Implementation of Alternating Direction Explicit Methods to higher dimensional Black-Scholes Equation, AIP Conf. Proc. 1773, 030001; 2016*

Book chapter:

- *Z. Bučková, B. Stehlíková, D. Ševčovič: Numerical and analytical methods for bond pricing in short rate convergence models of interest rates, book chapter in the book Advances in Mathematics Research. Volume 21, 2016*

Other published work or preprints:

- *Z. Bučková, J. Silva, M. Ehrhardt, M. Günther: STRIKE Novel Methods in Computational Finance, A European mathematical research training network, ECMI Newsletter 55, March 2014*
- *Z. Bučková, M. Ehrhardt: Splitting Methods on Special Meshes, ECMI Newsletter 56, October 2014*
- *A. Zocca, Z. Bučková, I. G. Minelli, M. Gastaldello, M. Aleandri, A. Trucchia, D. V. Greetham, A. Guterman, P. Giavedoni, A. Tsipenyuk, D. Cusseddu, R. B. Lijó, Z. Varbanov, B. Zlatanovska, A. Stojanova, M. Kocaleva, D. Bikov, A. Melchiori, A. Sgalambro: Mathematical and statistical analyses to support service performance forecasting in queuing systems, Study Groups with Industry, Problem presented by QURAMI company at 124th ESGI, preprint, 2016*

Abstracts from international and domestic conferences:

- Z. Bučková, I. Tsukerman, M. Ehrhardt: Traditional vs. Trefftz Difference Schemes for the Black-Scholes Equation, extended abstract, AMiTaNS conference 2016
- Z. Bučková (Zíková), Three-factor convergence model of interest rate, MMEI 2012 : Joint Czech-German-Slovak Conference, 2012 p. 21
- J. Halgašová, B. Stehlíková, Z. Bučková (Zíková), Three-factor convergence model of interest rate, ISCAMI 2012, Ostrava : Universitas Ostraviensis, 2012 p. 40-41
- Z. Bučková (Zíková), Convergence model of interest rate, Študentská vedecká konferencia FMFI UK, Bratislava 2011 : Zborník príspevkov, Bratislava : Fakulta matematiky, fyziky a informatiky UK, 2011 p. 70

8 Grants

- VEGA 1/0747/12: Kvalitatívna a kvantitatívna analýza parabolických parciálnych diferenciálnych rovníc a ich aplikácie
- Marie Curie International Training Network (ITN, 01/2013 - 12/2016), FP7-PEOPLE-2012-ITN (FP7 Marie Curie Action, Project Multi-ITN STRIKE - Novel Methods in Computational Finance
- APVV-14-0069, Modern methods for solving nonlinear partial differential equations in financial mathematics
- DAAD 01/2013-12/2014, NL-BS-AO: bilateral German-Slovakian Project Numerical Solution of nonlinear Black–Scholes sequation
- VEGA 1/0251/16: Kvantitatívna analýza modelov úrokových mier v podmienkach eurozóny a pristupujúcich krajín a jej aplikácie

9 Teaching activities

- Exercises to the lectures: financial derivatives (UK), PDE (UK), microeconomy (UK), ODE (BUW), PDE (BUW), financial mathematics (BUW)
- Supervision of 2 bachelor thesis has been done. Currently 1 bachelor thesis on UK and 1 master thesis on BUW is in progress.

10 List of given talks at international and domestic conferences

1. Z. Bučková (Zíková): “Models of interest rate”, ISCAMI 2012: 13th International Student Conference on Applied Mathematics and Informatics, International conference, Malenovice, May 10-13, 2012
2. Z. Bučková (Zíková), B. Stehlíková: “Models of interest rate, Three factor convergence model”, MMEI 2012: 17th International Conference on Mathematical Methods in Economy and Industry, International conference, Berlin, 24. -28. June 2012

3. EAPG workshop 2012, Stretnutie mladých ekonómov
4. Z. Bučková (Zíková), B. Stehlíková, J. Halgašová: "Estimating the short rate from the termstructures in the Vasicek model ", ALGORITMY 2012: Conference on Scientific computing. Vysoké Tatry. Podbanské. 9-14.9.2012
5. Z. Bučková (Zíková), B. Stehlíková, J. Halgašová: "Estimating the short rate from the termstructures in the Vasicek model ", ICCS 2013 Conference: International Conference on Computational Science, Barcelona, 5.- 7.June 2013
6. Z. Bučková, M. Ehrhardt, M. Günther, "Fichera theory and its application in finance", Minisymposium Computational Finance in the framework of ECMI 2014 - The 18th European Conference on Mathematics for Industry, Taormina, Siciliy, June 9-13.
7. Z. Bučková, M. Ehrhardt, M. Günther, "Numerical analysis of Alternating direction explicit methods for convection-reaction-diffusion equation", Sixth Conference Finite Difference Methods: Theory and Applications, Lozenetz, Bulgaria, June 18-23, 2014 (organized by Ruse)
8. Z. Bučková, M. Ehrhardt, M. Günther, "The Alternating Direction Explicit Method - Numerical analysis and its Application in Finance", Summer School on Computational Finance in the framework of Conference MMEI 2014 - Mathematical Methods in Economics and Industry, Bratislava -Smolenice castle, September 7-12, 2014
9. Z. Bučková: Mid-Term Review, ITN Strike, Würzburg, October 1, 2014
10. Z. Bučková, M. Ehrhardt, M. Günther, "Implementation of the Alternating Direction Explicit Methods in Multidimensional Models in Finance", SCF2015 Conference Stochastics & Computational Finance 2015 - From Academia to Industry, Lisbon, Portugal, July 6-10, 2015
11. Z. Bučková, M. Ehrhardt, M. Günther, "Implementation of ADE methods for higher-dimensional models ", ICCF2015 Conference International Conference on Computational Finance, Greenwich, UK, December 14-18, 2015
12. Z. Bučková, M. Ehrhardt, M. Günther, "Numerical analysis of Alternating direction explicit methods and its implementation for higher dimensional and nonlinear Black-Scholes model", Minisymposium on Financial Mathematics at ALGORITMY 2016 Conference, Podbanske, Slovakia, March 13-18, 2016
13. Z. Bučková, B. Stehlíková, "Modelling of interest rate", 6th International Conference, NAA 2016 - Numerical Analysis and Its Applications, Lozenetz, Bulgaria, June 15-20, 2016
14. Z. Bučková, M. Ehrhardt, I. Tsukerman, "Traditional vs. Trefftz Difference Schemes for the Black-Scholes Equation", special session Computational Finance; AMiTaNS-16, Albena, Bulgaria, June 22-27, 2016
15. Z. Bučková, M. Ehrhardt, M. Günther, "Advanced Numerical Methods in Finance for Black-Scholes model", special session Computational Finance; AMiTaNS-16, Albena, Bulgaria, June 22-27, 2016

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[3] Z. Bučková, M. Ehrhardt, and M. Günther. Alternating Direction Explicit Methods for Convection Diffusion Equations. *Acta Math. Univ. Comenianae*, Vol. LXXXI:309–325, 2015.

[4] Z. Bučková, M. Ehrhardt, and M. Günther. Fichera theory and its application to finance. *Proceedings ECMI 2014, Taormina, Sicily, Italy*, 2016.

[5] Z. Bučková, J. Halgašová, and B. Stehlíková. Short rate as a sum of two CKLS-type processes. *accepted for publication in Proceedings of Numerical analysis and applications conference, Springer Verlag in LNCS*, 2016.

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