Abstract

In this thesis, we deal with two problems of extremal graph theory: The Cage Problem and The Expander Graphs Problem. In *The Cage Problem*, we look for every $k \geq 3$ and $g \geq 3$ a (k, g)-graph (i.e., a k-regular graph of girth g) of the least possible order, called a *cage*. *The Expander Graphs Problem*, on the other hand, involves the problem of constructing families of highly connected but at the same time sparse graphs.

In this paper, we investigate the construction techniques and structural properties of (k, g)-graphs and cages, with a special focus on the need for recursive construction of $(k, g+1)$-graphs from (k, g)-graphs. We show that such a recursive construct, which would result in larger and larger graphs with a constant multiple coefficient, is not possible for even g. Nevertheless, we introduce a new recursive method for constructing $(k+1, 6)$-graphs from $(k, 6)$-graphs.

When investigating the structural properties of (k, g)-graphs and cages, we consider a broader perspective by examining the spectra of the orders of (k, g)-graphs, which are defined as sets of all possible orders of (k, g)-graphs for given parameters k and g. Using both theoretical and computational approaches, we determine complete spectra of orders for different pairs (k, g) and present partial results for parameters for which the spectra remain incomplete for the time being.

Next, we study $(k, g, \underline{g+1})$-graphs, which are (k, g)-graphs that do not contain cycles of length $g+1$. We present a new lower estimate on the orders of these graphs and identify several $(k, g, \underline{g+1})$-cages. Next, we prove Campbell's hypothesis about the uniqueness of the smallest $(3, 6, \underline{7})$-graph with an odd circumference of 11.

In conclusion, we examine the problem of the existence of expanding graphs and examine the potential connections between cages and expanding graphs. We present a possible connection between cages and expanding graphs by introducing three interrelated variants of the Bermond and Bollobás hypothesis, originally formulated in 1981 in the context of the Degrees/Diameter Problem. We adapt the original hypothesis to cages, with the most robust variant formulated as follows: *Does there exist a constant c such that for every pair of parameters k, g there exists a k-regular graph of girth g and order not exceeding $M(k, g)+c$?*; where $M(k, g)$ denotes the value of the so-called Moore bound for cages. We show that an affirmative answer to any of the three variants of Bermond and Bollobás's hypothesis formulated for the cages in this work would yield the existence of expander graphs (an infinite sequence of expanding graphs); Thus, this creates a fundamental link between the cages and the expanding graphs.