Block colorings of designs

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A Steiner system $S(2, k, v)$ is a design $(V, \mathcal{B})$ where V is a set of v *points* and $\mathcal{B}$ is a collection of k-element subsets of V called *blocks* such that every 2-element subset of V is contained in exactly one block. In a system $(V, \mathcal{B})$, any set of pairwise disjoint blocks makes a *partial parallel class*. A partial parallel class that partitions V is a *parallel class*; if a partial parallel class partitions $V \setminus \{x\}$ for some point x then it is called an *almost parallel class*.

A *proper block coloring* of a design $(V, \mathcal{B})$ is a mapping $c : \mathcal{B} \mapsto C$ such that for any two blocks $B, B' \in \mathcal{B}$, if $c(B) = c(B')$ then $B \cap B' = \emptyset$. The minimum number of colors needed to color blocks in $(V, \mathcal{B})$ is called its *chromatic index*.

Basic results and open problems in proper block colorings of designs will be discussed, mostly in the case of Steiner systems $S(2, k, v)$ for $k = 3$ (i.e. *Steiner triple systems*) and for $k = 4$. In particular, classes of systems for which the chromatic index attains its minimum will be presented.

Special attention will be given to projective triple systems. Let W_m be an $(m+1)$-dimensional vector space over $\mathbb{F}_2$ and $\oplus$ be an operation of vector addition in W_m (performed modulo 2 componentwise). A *projective triple system* $(V, \mathcal{B})$ is a Steiner triple system in which each point in V is represented by a non-zero vector in W_m and every two distinct points, corresponding to vectors $\mathbf{x}$ and $\mathbf{y}$, define a unique triple formed by $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \oplus \mathbf{y}\}$. The exact value for the chromatic index of projective triple systems will be determined, together with a polynomial time algorithm to obtain such a coloring.

In relation to proper block coloring, main results on the existence of parallel classes and almost parallel classes in Steiner systems $S(2, k, v)$ will be discussed.