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Fakulta matematiky, fyziky a informatiky

Mária Holecyová

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MAXIMUM PRINCIPLE FOR INFINITE HORIZON DISCRETE TIME OPTIMAL CONTROL PROBLEMS

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Predkladateľ: Mgr. Mária Holecyová
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky Univerzity Komenského
Mlynská dolina
842 48 Bratislava

Školiteľ: prof. RNDr. Pavel Brunovský, DrSc.
Katedra aplikovanej matematiky a štatistiky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského v Bratislave

Študijný program: 9.1.9 Aplikovaná matematika

Predseda odborovej komisie:

prof. RNDr. Marek Fila, DrSc.
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského
Mlynská Dolina, 842 48 Bratislava

1 Introduction

In economic optimal control models as well as in physics [2], engineering [3] and many other fields it is often impossible to predict the length of the time horizon. Therefore an objective function is formulated on infinite horizon and especially in economic models it is discounted. The discount ensures that the effect of the solutions to the objective function decreases with passing time which solves the dilemma of setting the length of the horizon as well as the final state.

We assume that response $\{x_t\}_{t=0}^{\infty}$ and control $\{u_t\}_{t=0}^{\infty}$ are bounded sequences, but do not vanish in infinity necessarily. Hence, the problem we consider is of the following form:

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} \delta^t f_t^0(x_t, u_t) \to \max \tag{1.1}$$

$$x_{t+1} = F_t(x_t, u_t) \quad \text{for all } t \in \mathbb{N}_0 \tag{1.2}$$

$$x_0 = \bar{x}, \tag{1.3}$$

where $\bar{x}$ and the discount $\delta \in (0, 1)$ are given, $x_t \in X \subset \mathbb{R}^n$, $u_t \in U \subset \mathbb{R}^m$, U open. We denote $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$, $\mathbf{u} = \{u_t\}_{t=0}^{\infty}$ and assume $f_t^0 \in C^1(X \times U, \mathbb{R})$ for all $t \in \mathbb{N}_0$ $F_t \in C^1(X \times U, X)$ for all $t \in \mathbb{N}_0$. We call J objective function, F_t dynamics. x_t state variable and u_t control variable. We assume $(\mathbf{x}, \mathbf{u}) \in \ell_\infty^n \times \ell_\infty^m$.

Although we consider discrete-time problems we focus on establishing neccesary conditions of optimality in the spirit of Pontryagin maximum principle which was originally devoloped for continuous-time models [1].

While for the continuous-time setting Pontryagin maximum principle can be easily adapted for a wide class of problems, this is not the case of the discrete-time problems unless extra convexity conditions are imposed. So instead of the maximum condition we strive for a necessary condition of this maximum with less restrictive assumptions.

The current research on this topic is not rich. We found only two articles closely related to our problem. In the first Blot and Chebbi [4] solved it as limit case of finite horizon problem. In continous framework, the extension from finite to infinite horizon can be obtained without any restrictions due to its invertible dynamics. However, in discrete time invertibility is not ensured and therefore it has to be formulated as an additional condition. So Blot and Chebbi established the maximum principle in the space ℓ_1 with cost function under the condition that $A_t = D_{x_t}F_t(\hat{x}_t, \hat{u}_t)$ are invertible for all t. Later, Blot and Hayek [5] considered the same problem as we did and via tools of functional analysis formulated condition

$$\sup_{t\in\mathbb{N}_0} \|A_t\|_\infty < 1.$$

We adapt their approach, but while their results are based on Ioffe-Tihomirov theorem, we employ the closed range theorem.

Theorem 1. ***Closed range theorem***
Let X, Y be Banach spaces and T a closed linear operator from X to Y. Then the following propositions are equivalent:

1. $\mathcal{R}(T)$ *is closed*
2. $\mathcal{R}(T^*)$ *is closed*
3. $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp = \{y \in Y : \langle y^*, y\rangle = 0 \text{ for all } y^* \in \mathcal{N}(T^*)\}$
4. $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in \mathcal{N}(T)\}$.

Proof. The proof can be found in [6].a □

This idea first appeared in diploma thesis by Beran [7], where its reduced form $2 \Rightarrow 4$ is applied, but it was developed for non-discounted cost functions and the research under which conditions it can be applied was incomplete.

We employ its reduced form $1 \Rightarrow 4$ which allows us to establish pseudo-Potryagin principle however condition of closed range has to satisfied. Therefore we develop the theory of exponential dichotomy for linear difference equations [8], [9] and derive conditions under which the maximum principle hold.

2 The Proposed Method

In this section we desribe the method establishing the pseudo-Pontryagin maximum principle for discrete-time optimal control problems on infinite horizon. At first we consider the problem with linear autonomous dynamics, then we extend our results to general dynamics.

We apply a standard method of constructing perturbations along the optimal solutions. At first we have to ensure that the cost function is Fréchet differentiable. Therefore we formulate and prove the following proposition.

Proposition 1. The function $J : \ell_p^n \times \ell_p^n \to \mathbb{R}$, $p \in \langle 1, \infty \rangle$, defined by $J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} \delta^t f^0(x_t, u_t)$, where $x_t \in X \subset \mathbb{R}^n$, $u_t \in U \subset R^m$ and $f^0 \in C^1(X \times U, \mathbb{R})$ is Fréchet differentiable.

2.1 The Problem with Autonomous Linear Dynamics

We describe our method on the infinite-horizon discrete-time optimal control model with linear autonomous dynamics

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} \delta^t f^0(x_t, u_t) \to \max \tag{2.1}$$

$$x_{t+1} = Ax_t + Bu_t + d \quad \text{for all } t \in \mathbb{N}_0 \tag{2.2}$$

$$x_0 = \bar{x}, \tag{2.3}$$

given $\bar{x}$, $d \in \mathbb{R}^n$ $n \times n$ matrix A, $n \times m$ matrix B and discount $\delta \in (0, 1)$ are given. We denote $x_t \in \mathbb{R}^n = X$, $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$, $u_t \in \mathbb{R}^m = U$ $\mathbf{u} = \{u_t\}_{t=0}^{\infty}$, objective function $f^0 \in C^1(X \times U, \mathbb{R})$, $f \in C^1(X \times U, X)$.

Firstly, we construct perturbations along the optimal solution and then we formulate necessary conditions of optimality.

Let $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ be optimal solution of problem (2.1) - (2.3). A pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \ell_\infty^{n+m}$ is called admissible, if for all $\varepsilon > 0$ it holds

$$\begin{aligned} \hat{x}_0 + \varepsilon\alpha_0 &= \bar{x} \\ \hat{x}_{t+1} + \varepsilon\alpha_{t+1} &= A(\hat{x}_t + \varepsilon\alpha_t) + B(\hat{u}_t + \varepsilon\beta_t) + d \quad \text{for all } t \in \mathbb{N}_0, \end{aligned}$$

i. e. $\{\hat{x}_t + \varepsilon\alpha_t, \hat{u}_t + \varepsilon\beta_t\}$ satisfies (2.2) and (2.3).

Because of (2.3), one has $\alpha_0 = 0$. Next, we apply equation $\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + d$ and the system can be rewritten to

$$\begin{aligned} \alpha_0 &= 0 \\ \alpha_{t+1} &= A\alpha_t + B\beta_t \quad \text{for all } t \in \mathbb{N}_0. \end{aligned}$$

From the definition of an admissible vector, $J(\hat{\mathbf{x}}+\varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}}+\varepsilon\boldsymbol{\beta})$ cannot increase with ε (≥ 0) from the maximum. We have already shown that J is Fréchet differentiable in ℓ_∞^{n+m} (Proposition 1), therefore

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} \leq 0 \\ \frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} - \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} - \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} \leq 0 \end{aligned} \tag{2.4}$$

As $\frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} - \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} - \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} = -\frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0}$ (2.4) can be rewritten to

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} = \sum_{t=0}^{\infty} \delta^t [D_{x_t} f^0(\hat{x}_t, \hat{u}_t)\alpha_t + D_{u_t} f^0(\hat{x}_t, \hat{u}_t)\beta_t] \\ &= DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal. \end{aligned}$$

This notation can be simplified by defining $\mathbf{A}$, $\mathbf{B}$ and by introducing a vector of shift operator $\boldsymbol{\sigma}$, such that $(\mathbf{A}\boldsymbol{\alpha})_t = A\alpha_t$, $(\mathbf{B}\boldsymbol{\alpha})_t = B\beta_t$ and $(\boldsymbol{\sigma}\alpha)_t = \alpha_{t+1}$ and we obtain

$$\alpha_0 = 0 \tag{2.5}$$

$$(\boldsymbol{\sigma} - \mathbf{A})\boldsymbol{\alpha} - \mathbf{B}\boldsymbol{\beta} = (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0. \tag{2.6}$$

Remark 1. Let us define an operator $\pi_0 = (\pi_0^x, \mathbf{0})$ such that $\pi_0(\mathbf{x}, \mathbf{u})^\intercal = x_0$ and an operator $\mathcal{L} : \ell_\infty^n \times \ell_\infty^m \to \ell_\infty^n$, $\mathcal{L} = (\pi_0, (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B}))$. Then conditions (2.5) and (2.6) can be replaced by $\mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0$ or $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{N}(\mathcal{L})$.

In order to apply closed range theorem, $\mathcal{L}$ needs to be bounded.

Proposition 2. Let $\mathbf{A}$ and $\mathbf{B}$ be general linear operators. Then $\mathcal{L} : \ell_p^n \times \ell_p^m \to \ell_p^n$, $\mathcal{L} = (\pi_0, (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B}))$ is bounded linear operator for any $p \in \langle 0, \infty \rangle$.

Theorem 2. *(Necessary conditions of optimality) Assume that the operator* $\mathcal{L} = (\pi_0, \boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})$ *has closed range. Then* $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0$ *for all admissible* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *if only if there exists* $\boldsymbol{\phi} \in (\ell_\infty)^*$ *such that*

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \mathcal{L}^* \boldsymbol{\phi} \tag{2.7}$$

Moreover, if one has $\boldsymbol{\phi} = \boldsymbol{\psi} + \boldsymbol{\phi}^s$, *where* $\boldsymbol{\psi} = \{\psi_t\}_{t \in \mathbb{N}_0} \in \ell_1$ *and* $\boldsymbol{\phi}^s \in \ell_s$, *then* $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0$ *for all admissible* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *if*

$$\begin{aligned} D_{x_t} f^0(\hat{x}_t, \hat{u}_t) &= \psi_{t-1} - \delta A^* \psi_t \text{ for all } t \in \mathbb{N} \\ D_{u_t} f^0(\hat{x}_t, \hat{u}_t) &= -\delta B^* \psi_t \text{ for all } t \in \mathbb{N}_0. \end{aligned} \tag{2.8}$$

In order to prove that (2.7) can be rewritten to (2.8) we adapt the approach of Blot and Hayek [5] based on the following lemma

Lemma 1. *If* $\boldsymbol{\psi}^s \in \ell_s$, *then there exists* $k \in \mathbb{R}$ *such that for all* $\mathbf{x} \in c$, $\boldsymbol{\psi}^s \mathbf{x} = k \lim_{t \to \infty} x_t$.

Proof. The proof can be found in [10]. □

2.2 The Problem with General Dynamics

In this section, we replace the linear autonomous dynamics (2.2) by generalized dynamics $F_t \in C^1(X \times U, \mathbb{R})$ for all $t \in \mathbb{N}_0$, i.e. we consider the problem

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{\infty} \delta^t f^0(x_t, u_t) \to \max \tag{2.9}$$

$$x_{t+1} = F_t(x_t, u_t) \quad \text{for all } t \in \mathbb{N}_0 \tag{2.10}$$

$$x_0 = \bar{x}. \tag{2.11}$$

Denote

$$
\begin{aligned}
D_{x_t}F(\hat{x}_t, \hat{u}_t) &= A_t \quad \text{for all } t \in \mathbb{N}_0 \\
D_{u_t}F(\hat{x}_t, \hat{u}_t) &= B_t \quad \text{for all } t \in \mathbb{N}_0 \\
(A_0, A_1, ...) &= \mathbf{A} \\
(B_0, B_1, ...) &= \mathbf{B}
\end{aligned}
$$

$$
\mathcal{L} = (\pi_0, (\boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})), \quad \mathcal{L} : \ell_\infty^n \times \ell_\infty^m \to \ell_\infty^n.
$$

Again, we construct pertubations along the optimal solution, i.e. curves that start from the optimal solution, their directions are given and conditions (2.10), (2.11) are fulfilled.

Definition 1. We call a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \ell_\infty^n \times \ell_\infty^m$ an admissible vector if there exist $\varepsilon_0 > 0$ and differentiable curves $\mathbf{p}(\varepsilon_0) = \{p_t(\varepsilon_0)\}_{t\in\mathbb{N}_0}$, $\mathbf{q}(\varepsilon_0) = \{p_t(\varepsilon_0)\}_{t\in\mathbb{N}_0}$, where

$$
\begin{aligned}
p_t &: \langle 0, \varepsilon_0) \to X \\
q_t &: \langle 0, \varepsilon_0) \to U
\end{aligned}
$$

for all $t \in \mathbb{N}_0$ such that the following conditions hold

i) $\mathbf{p}(0) = \mathbf{q}(0) = 0$

ii) $\mathbf{p}'(0) = \boldsymbol{\alpha}$ and $\mathbf{q}'(0) = \boldsymbol{\beta}$

iii) for each $\varepsilon \in \langle 0, \varepsilon_0)$ and $t \in \mathbb{N}_0$

$$
\begin{aligned}
p_0(\varepsilon) &= 0 \\
\hat{x}_{t+1} + p_{t+1}(\varepsilon) &= F_t(\hat{x}_t + p_t(\varepsilon), \hat{u}_t + q_t(\varepsilon))
\end{aligned}
$$

and $(\hat{\mathbf{x}} + \mathbf{p}(\varepsilon), \hat{\mathbf{u}} + \mathbf{q}(\varepsilon)) \in \ell_\infty^n \times \ell_\infty^m$.

If in any direction there exist an admissible perturbation curve, we can use it to derive the necessary conditions of optimality as in the case in linear autonomous dynamics. In the following proposition we state under which conditions this is the case.

Proposition 3. Assume that $\mathcal{L}$ has a closed complement to its null space. Then each vector $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{N}(\mathcal{L})$ is admissible.

Next, we proceed as in the problem with linear autonomous dynamics. If $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible and $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is an optimal solution then

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} &\leq 0 \\ \frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} - \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} - \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} = -\frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} &\leq 0 \end{aligned} \tag{2.12}$$

Hence, again we have

$$\begin{aligned} 0 = \frac{\partial}{\partial \varepsilon} J(\hat{\mathbf{x}} + \varepsilon\boldsymbol{\alpha}, \hat{\mathbf{u}} + \varepsilon\boldsymbol{\beta})|_{\varepsilon=0} &= \sum_{t=0}^{\infty} \delta^t [D_{x_t} f^0(\hat{x}_t, \hat{u}_t)\alpha_t + D_{u_t} f^0(\hat{x}_t, \hat{u}_t)\beta_t] \\ &= DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal. \end{aligned}$$

and the necessary conditions are analogous.

Theorem 3. *(Necessary conditions of optimality) Assume that the operator* $\mathcal{L} = (\pi_0, \boldsymbol{\sigma} - \mathbf{A}, -\mathbf{B})$ *has closed range. Then* $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0$ *for all admissible* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *if only if there exists* $\boldsymbol{\phi} \in (\ell_\infty)^*$ *such that*

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) = \mathcal{L}^* \boldsymbol{\phi} \tag{2.13}$$

Moreover, if one has $\boldsymbol{\phi} = \boldsymbol{\psi} + \boldsymbol{\phi}^s$*, where* $\boldsymbol{\psi} = \{\psi_t\}_{t \in \mathbb{N}_0} \in \ell_1$ *and* $\boldsymbol{\phi}^s \in \ell_s$*, then* $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\boldsymbol{\alpha}, \boldsymbol{\beta})^\intercal = 0$ *for all admissible* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *if*

$$\begin{aligned} D_{x_t} f^0(\hat{x}_t, \hat{u}_t) &= \psi_{t-1} - \delta A_t^* \psi_t \text{ for all } t \in \mathbb{N} \\ D_{u_t} f^0(\hat{x}_t, \hat{u}_t) &= -\delta B_t^* \psi_t \text{ for all } t \in \mathbb{N}_0. \end{aligned} \tag{2.14}$$

3 Closed Range of $\mathcal{L}$

In the previous chapter, we assumed that the operator $\mathcal{L} : \ell_\infty^n \times \ell_\infty^m \to \ell_\infty^n$, $\mathcal{L} = (\pi_0, (\sigma - \mathbf{A}, -\mathbf{B}))$ has closed range and that the complement to its null space exists and is closed as well. Now we show under which conditions this is the case. At first we explore the autonomous system, i.e. where matrices A_t, B_t are constant for any t, then we derive conditions for the nonautonomous system. In both cases we consider $\mathcal{L} : \ell_p^n \times \ell_p^m \to \ell_p^n$, $p \in \langle 1, \infty \rangle$.

Definition 2. (Exponential Dichotomy)

Consider a linear difference equation

$$v_{t+1} = A_t v_t + B_t w_t + z_t, \tag{3.1}$$

with an initial condition $v_0 = z_0$ where $\mathbf{v} = \{v_t\}_{t\in\mathbb{N}} \in \ell_p^n$, $\mathbf{w} = \{w_t\}_{t\in\mathbb{N}} \in \ell_p^m$ $p \in \langle 1, \infty \rangle$, $t \in \mathbb{N}$ and A_t are $n \times n$ matrices. We say that the linear difference equation (3.1) has an exponential dichotomy on $\mathbb{N}$ if there exist $C \geq 1$, $\lambda \in (0, 1)$ and a family of projections $P_t, t \in \mathbb{N}$ such that

1. $P_{t+1}A_t = A_t P_t$, i.e. they commute,

2.
$$\|P_t\| \leq C \tag{3.2}$$
$$\left\| \prod_{i=t-1}^{s} A_j^- \right\| \leq C\lambda^{t-s} \text{ for all } t \geq s, \tag{3.3}$$

   where $A_t^- = P_{t+1}A_t|_{\mathcal{R}(P_t)}$,

3. $A_j^+ = (I - P_{t+1})A_t|_{\mathcal{R}(I-P_t)} = Q_{t+1}A_t|_{\mathcal{R}(Q_t)}$ are invertible for all $t \in \mathbb{N}$ and

$$\left\| \prod_{i=t}^{s-1} (A_j^+)^{-1} \right\| \leq C\lambda^{s-t}, \text{ for all } t < s. \tag{3.4}$$

For the sake of simplicity, let us denote

$$\Psi(t,s) = \begin{cases} \prod_{i=t-1}^{s} A_j^-, & \text{if } t \geq s \\ \prod_{i=t}^{s-1} (A_j^+)^{-1} & \text{if } t < s. \end{cases}$$

so that equations (3.3), (3.4) can be rewritten to

$$\|\Psi(t,s)\| \leq C\lambda^{|t-s|}.$$

Theorem 4. *Let the linear difference equation (3.1) have an exponential dichotomy on* $\mathbb{N}$ *and* B_t *be bounded for all* t*. Then the operator* $\mathcal{L}$ *has closed range and complement to its null space exists and is closed.*

3.1 Special Cases

In the last section, we explore the cases in which exponential dichotomy can be effectively verified and on a simple example we show that without dichotomy closed range probably may not hold.

Proposition 4. If the eigenvalues of A do not lie on the unit circle, there exists a projection matrix P, such that $PA|_{\mathcal{R}(P)}$ has eigenvalues outside the unit circle, $(I-P)A|_{\mathcal{R}(I-P)}$ has eigenvalues inside the unit circle, hence it is regular. Moreover, there exist $C \geq 1$ and $\lambda \in (0,1)$ such that

1. $\|\Psi(t,s)\| < C\lambda^{t-s}$, for all $t \geq s$,
2. $\|\Psi(t,s)\| < C\lambda^{s-t}$, for all $t < s$,

where

$$\Psi(t,s) = \begin{cases} (PA|_{\mathcal{R}(P)})^{t-s}, & \text{if } t \geq s \\ ((I-P)A|_{\mathcal{R}(I-P)})^{-(s-t)} & \text{if } t < s. \end{cases}$$

Corollary 1. If A_t is constant for all t and have no eigenvalues on the unit circle, then the linear difference equation (3.1) has an exponential dichotomy on $\mathbb{N}$.

Proposition 5. If matrices A_t converge to a matrix A_∞ such that its eigenvalues do not lie on the unit circle, the linear difference equation (3.1) has an exponential dichotomy on $\mathbb{N}$.

Proposition 6. If matrices A_t are periodic with period T and the matrix $\mathcal{A} = A_T A_{T-1} \dots A_1$ has no eigenvalues on unit cirle and it is regular, then the linear difference equation (3.1) has exponential dichotomy on $\mathbb{N}$.

Proposition 7. If the linear difference equation (3.1) has an exponential dichotomy on $\mathbb{N} \backslash K$, where $K = \{1, \dots, T\}$, then it has exponential dichotomy on $\mathbb{N}$.

While the condition of exponential dichotomy extends the current framework of problems for which the maximum principle holds, we could not formulate it as an equivalence. However, we can find an example when exponential dichotomy is not satisfied and the range of $\mathcal{L}$ is not closed, hence the closed range theorem cannot be applied.

Example 1. We assume linear autonomous dynamics and $A = 1$, $B = 0$, so that the state can be rewritten as $x_{t+1} = x_t$. Then

$$\mathcal{R}(\mathcal{L}) = \{\mathbf{z} \in \ell^1_\infty : z_t = x_{t+1} - x_t, \mathbf{x} \in \ell^1_p\}.$$

For a given $\varepsilon > 0$, we choose $\mathbf{z}^\varepsilon$ such that $z^\varepsilon_0 = 0$ and $z^\varepsilon_t = t^{-(1+\varepsilon)}$ for $t \geq 1$. Then corresponding $\mathbf{x}^\varepsilon$ is given as $x^\varepsilon_0 = 0$ and $x^\varepsilon_t = \sum_{s=2}^{t-1}(s-1)^{-(1+\varepsilon)}$, $t \geq 1$. So while $\mathbf{z}^\varepsilon, \mathbf{x}^\varepsilon \in \ell^1_\infty$ and $\lim\limits_{\varepsilon \to 0} z_t{}^\varepsilon = t^{-1}$, so $\lim\limits_{\varepsilon \to 0} \mathbf{z}^\varepsilon \in \ell^1_\infty$, but $\lim\limits_{\varepsilon \to 0} \mathbf{x}^\varepsilon \notin \ell^1_\infty$. Therefore $\mathcal{R}(\mathcal{L})$ is not closed.

Summary and Conclusions

We focused on infinite-horizon, discrete-time optimal control problems and established necessary conditions of optimality in the sense of Potryagin maximum principle. We considered problems with linear autonomous state equation $x_{t+1} = Ax_t + Bu_t + d$ and general state $x_{t+1} = F_t(x_t, u_t)$.

We employed direct approach via tools of functional analysis rather than reduction to finite horizon and faced four main challenges. At first, we had to prove that the cost function is Fréchet differentiable. Secondly, we derived necessary conditions of optimality with adjoint variable belonging to the dual space of ℓ_∞, $(\ell_\infty)^* = \ell_1 \oplus \ell_s$. We managed to circumvent its non-sequential component ℓ_s. The most significant results are discribed in the last section where we formulate assumptions under which the necessary conditions hold, i.e. when the operator $\mathcal{L}$ has closed range. Moreover, in the case of general dynamics we had to show that its null space is complemented and the complement is closed.

Generally, the condition is formulated as an exponential dichotomy on $\mathbb{N}$, i.e. there exist $C \geq 1$, $\lambda \in (0,1)$ and bounded projection matrices P_t such that

$$\|\Psi(t,s)\| \leq C\lambda^{|t-s|}, \quad \text{for any } t,s \in \mathbb{N}_0,$$

where

$$\Psi(t,s) = \begin{cases} \prod_{i=t-1}^{s} P_{t+1}A_t|_{\mathcal{R}(P_t)}, & \text{if } t \geq s \\ \prod_{i=t}^{s-1}(I-P_{t+1})A_t^{-1}|_{\mathcal{R}(I-P_t)} & \text{if } t < s. \end{cases}$$

Next, we proved that the system possesses exponential dichotomy if

- $A_t = A$ are constant and A has no eigenvalues on the unit circle
- matrices A_t converge to a matrix A_∞ such that its eigenvalues do not lie on the unit circle,
- matrices A_t are periodical with period T and the matrix $\mathcal{A} = A_T A_{T-1} \dots A_1$ has no eigenvalues on the unit cirle and it is regular
- the linear difference equation (3.1) has an exponential dichotomy on $\mathbb{N}\backslash K$, where $K = \{1, \dots, T\}$, $T < \infty$.

There is definitely still a lot of space for future development of the presented framework. Moreover, further research can also be conducted in order to examine necessity of condition of exponential dichotomy. So far, we have managed to find an example where exponential dichotomy is not satisfied and the closed range theorem cannot be applied.

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