UNIVERZITA KOMENSKÉHO
V BRATISLAVE

FAKULTA MATEMATIKY, FYZIKY
A INFORMATIKY

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Autoreferát dizertačnej práce

SINGULARITY EVOLÚT A PEDÁLNYCH KRIVIEK V EUKLIDOVSKEJ A V MIKOVSKÉHO ROVINE

na získanie akademického titulu philosophiae doctor
v odbore doktorandského štúdia

9.1.7 Geometria a topológia

Bratislava 2013

Dizertačná práca bola vypracovaná v dennej forme doktorandského štúdia na Katedre algebry, geometrie a didaktiky matematiky Fakulty matematiky, fyziky a informatiky Univerzity Komenského v Bratislave.

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Obhajoba dizertačnej práce sa koná .... o .... h
pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie ....

9.1.7 Geometria a topológia

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Singularities of Evolutes and Pedal curves in the Euclidean and in the Minkowski plane

1 Introduction

Singularities of curves are an old field of research in mathematics. Some curves with singular points appear in ancient Greece geometers works and later became a field of study of many famous geometers who built a theory for their analysis and classification. Today, the study of singularities is an active field of research in different branches of mathematics such as differential geometry, algebra or topology.

In classical differential geometry, the use of the Euclidean plane as environment is most widespread (see e.g. [7],[6], [10]). In recent years, the Minkowski plane and its attributes occur in many publication dealing with curves (see e.g. [4], [8], [9]). We work with plane curves simultaneously in both planes. The thesis provides a description of basic concepts and notions of differential geometry of plane curves in mentioned planes. We report basic affine and metric characteristics of plane curves.

We study and classify singular points of two types of associated curves - evolutes and pedal curves using differential geometry. Associated curves are such curves which are generated by a geometrical construction from a given base curve. It is natural to study relationship between the base curve and the associated one. Singular points of evolutes and pedal curves are influenced by properties of the base curve. We provide full classification of singularities of evolutes and pedal curves and we point out their connection with particular points on the base curve. Achieved results are visualized by two visualization tools created for the purpose of our theoretical research.

2 Goals and tasks

The goals of the dissertation thesis were presented in the Project of dissertation (2012). One of the goals was to explore osculating circles in the Minkowski plane and to investigate their contact with the corresponding curve at vertices of finite order. The main goals were to study chosen curves associated with a given base curve in the Euclidean and in the Minkowski plane. The first task was to discuss singularities of these curves and to provide their full classification in both planes. The second task was to prepare visualization tools which enable to visualize achieved results.

We claim that the goals were partly fulfilled. We studied and classified singular points of evolutes and pedal curves in all details. Singularities of other associated curves were not discussed because of the lack of time, these shall be subjects of next research.

3 Differential geometry of curves

Throughout the thesis we discuss the geometry of both planes simultaneously. The affine geometry in the Minkowski plane is the same as in the Euclidean plane. Thus we can freely make use of all affine concepts and relations in the Minkowski plane as points, vectors, straight lines, half-planes, affine coordinates, orientation etc.. The basic difference between the two planes is that we consider the pseudo-scalar product in the Minkowski plane instead of standard scalar product used in the Euclidean plane. The change of the scalar product causes modifications in concepts related with it. In these concepts, the only difference is the sign change when considering one of the planes. Therefore, we use the symbol $\mathcal{E} \in \{-1, 1\}$ indicating this sign difference. $\mathcal{E} = -1$ for the Euclidean plane and $\mathcal{E} = 1$ for the Minkowski plane.

3.1 Geometry of the Euclidean and the Minkowski plane

Under *the Euclidean plane* $\mathbb{E}^2$ we mean the real affine plane $\mathbb{A}^2$ on vectors of which a bilinear quadratic form q of signature $(2, 0)$ is defined. *The Minkowski plane* $\mathbb{E}^{1,1}$ is the real affine plane $\mathbb{A}^2$ furnished with a bilinear quadratic form q of signature $(1, 1)$ on vectors of $\mathbb{A}^2$.

The polar form $\langle \mathbf{x}, \mathbf{y} \rangle$ corresponding to quadratic form $q(\mathbf{x})$ expressed as $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\left[q\left(\mathbf{x}+\mathbf{y}\right) - q\left(\mathbf{x}-\mathbf{y}\right)\right]$ will be called *the scalar product*. It holds $\langle \mathbf{x}, \mathbf{x} \rangle = q(\mathbf{x})$.

For any vector $\mathbf{x}$ we define its *sign* as $\operatorname{sgn} \mathbf{x} = \operatorname{sgn} q(\mathbf{x})$ and its *length* as $|\mathbf{x}| = |q(\mathbf{x})|^{\frac{1}{2}}$. For *a unit vector* it holds $|\langle \mathbf{x}, \mathbf{x} \rangle| = 1$, i.e. $\langle \mathbf{x}, \mathbf{x} \rangle = \pm 1$. Vectors $\mathbf{x}$ and $\mathbf{y}$ are *perpendicular* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Perpendicularity of vectors is denoted by $\mathbf{x} \perp \mathbf{y}$.

Vectors $\mathbf{e_1}, \mathbf{e_2}$ create *an orthonormal basis* if they are orthonormal, i.e. if $\langle \mathbf{e_1}, \mathbf{e_1} \rangle = q(\mathbf{e_1}) = \operatorname{sgn} \mathbf{e_1}$, $\langle \mathbf{e_1}, \mathbf{e_2} \rangle = 0$, $\langle \mathbf{e_2}, \mathbf{e_2} \rangle = q(\mathbf{e_2}) = \operatorname{sgn} \mathbf{e_2}$. *A Cartesian coordinate system* is an affine and orthonormal coordinate system $O, \mathbf{e_1}, \mathbf{e_2}$ with property $\operatorname{sgn} \mathbf{e}_1 = 1$. We work in a fixed Cartesian coordinate system in the considered plane.

Coordinate expressions of quadratic form $q(\mathbf{x})$ and scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ in any Cartesian coordinate system are $q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 - \mathcal{E} x_2^2$, $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - \mathcal{E} x_2 y_2$.

We distinguish three types of vectors in the plane. Vector $\mathbf{x}$ is *space-like* or *time-like* or *light-like* if $\operatorname{sgn} \mathbf{x} = 1$ or $\operatorname{sgn} \mathbf{x} = -1$ or $\operatorname{sgn} \mathbf{x} = 0$, respectively.

We define two operators which work on vectors. *A perpendicularity operator* $\mathbf{x} \to \mathbf{x}^{\perp}$ assigns the vector $\mathbf{x}^{\perp} = \operatorname{sgn} \mathbf{x}(\mathcal{E}x_2, x_1)$ to a vector $\mathbf{x} = (x_1, x_2)$. The perpendicularity converts type of vectors in the Minkowski plane: a non-zero vector perpendicular to a space-like vector is time-like and vice versa. In the Minkowski plane we define *a symmetry operator* $\mathbf{x} \to \mathcal{S}\mathbf{x}$ assigning the vector $\mathcal{S}\mathbf{x} = (x_2, x_1)$ to a vector $\mathbf{x} = (x_1, x_2)$. A vector $\mathbf{x}$ in the Minkowski plane is light-like $\Longleftrightarrow \mathcal{S}\mathbf{x} = \gamma\mathbf{x}$, $\gamma \in \{-1, 1\}$.

3.2 Plane curves

A singular point of a point function $P(t)$ is a point $P(t_0)$ such that $P^{(1)}(t_0) = \mathbf{0}$. Non-singular points are called *regular points.* We accept only isolated singular points of point functions. Singular points are often called *singularities* of point functions.

A point $P(t_0)$ is *a singular point of order* n, $n \geq 1$ of a point function $P(t)$ if $P^{(1)}(t_0) = \mathbf{0}, \ldots, P^{(n)}(t_0) = \mathbf{0}$. A point $P(t_0)$ is *a singular point of order exactly* n, $n \geq 1$ if $P^{(1)}(t_0) = \mathbf{0}, \ldots, P^{(n)}(t_0) = \mathbf{0}$ and $P^{(n+1)}(t_0) \neq \mathbf{0}$.

A parametrized curve in a plane is a smooth point function of a single real variable $P : I \to \mathbb{A}^2$, $I \subseteq \mathbb{R}$ which has only isolated singular points. *A regular parametrized curve* in a plane is a parametrized curve $P : I \to \mathbb{A}^2$, $I \subseteq \mathbb{R}$ without singular points.

A function $\varphi : J \to I$ is *a change of parameter* if φ is smooth, surjective and regular. A change of parameter has an inverse which is a change of parameter, too. Two parametrized curves $P : I \to \mathbb{A}^2$ and $Q : J \to \mathbb{A}^2$ are *equivalent* if there exists a change of parameter $\varphi : J \to I$ with $Q\,(u) = P\,(\varphi\,(u))$ for all parameters $u \in J$.

A (non-parametrized) *curve* in a plane is an equivalence class of parametrized curves with respect to change of parameter. *A parametrization of a curve* is one representative of the curve. Two parametrized curves equivalent each to other have the same set of points, so we can speak about points of a (non-parametrized) curve. Singularity or regularity of a point does not change after a reparametrization of a curve. Therefore these are attributes of a (non-parametrized) curve. So is the order of singular points.

The tangent line $p_P(t_0)$ to a curve $P(t)$ at a regular point $P(t_0)$ is the line through $P(t_0)$ in the direction of the vector $P^{(1)}(t_0)$. *The tangent line* to the curve $P(t)$ at a singular point $P(t_0)$ of order exactly n is the line through $P(t_0)$ in the direction of the vector $P^{(n+1)}(t_0)$. The vector $P^{(1)}(t_0)$ at a regular point or the vector $P^{(n+1)}(t_0)$ at a singular point is called the tangent vector of the curve $P(t)$ at the point $P(t_0)$.

The forward tangent half-line of a curve $P(t)$, $t \in I$ at point $P(t_0)$ is the ray $P(t_0)\mathbf{x}_+(t_0)$ determined by the vector $\mathbf{x}_+(t_0) = \lim_{t \to t_0+} \frac{P(t)-P(t_0)}{|P(t)-P(t_0)|}$. *The backward tangent half-line* $P(t_0)\mathbf{x}_-(t_0)$ is defined analogously. These tangent half-lines exist at a singular point $P(t_0)$ of finite order. If the singularity order is exactly n then tangent half-lines are determined by vectors $P^{(n+1)}(t_0+)$ and $(-1)^n P^{(n+1)}(t_0-)$.

3.3 Classification of singular points

We distinguish three types of singular points in differential geometry. A singular point $P(t_0)$ of a curve in which both the forward and the backward tangent half-lines are defined is called
(1) *a point of fraction* if the two half-lines do not lie on one straight line,
(2) *a cusp* if the half-lines coincide,
(3) *an insignificantly singular point* if the half-lines are opposite to each other.

A cusp $P(t_0)$ of a curve is *a cusp of the first kind* if the curve lies in both of the half-planes defined by the tangent line $p_P(t_0)$ for parameters sufficiently close to t_0. A cusp $P(t_0)$ of a curve is *a cusp of the second kind* if the curve lies in one of the half-planes defined by the tangent line $p_P(t_0)$ for parameters sufficiently close to t_0.

Let $P(t_0)$ be a singular point of order exactly n, $n \geq 1$ of a curve $P(t)$. Then it holds:
(1) If n is even then the point $P(t_0)$ is an insignificantly singular point of the curve.
(2) If n is odd then the point $P(t_0)$ is a cusp of the curve.
(3) Let n be odd and let l be the smallest natural number so that $P^{(n+1)}(t_0)$ and $P^{(m)}(t_0)$ are linearly independent. Then the point $P(t_0)$ is a cusp of the first kind for m odd and it is a cusp of the second kind for m even.

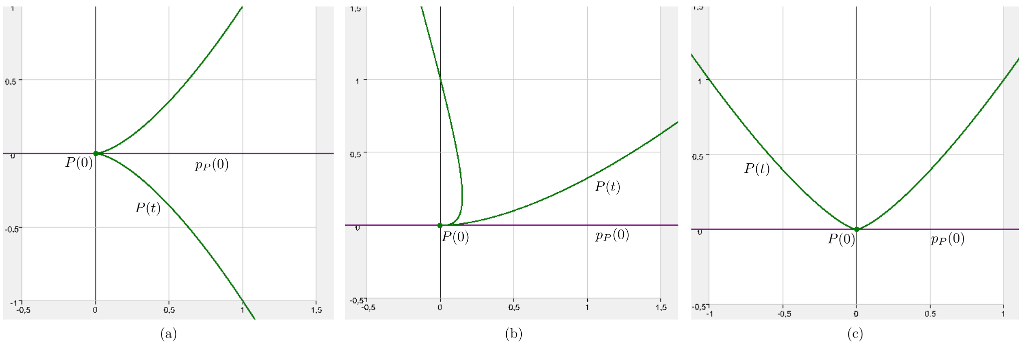

Figure 1: Singular points $P(0)$ and tangent lines at singular points of curves (a) $P(t) = (t^2, t^3)$ - $P(0)$ is a cusp of the first kind (b) $P(t) = (t^2 + t^3, t^4)$ - $P(0)$ is a cusp of the second kind (c) $P(t) = (t^3, t^4)$ - $P(0)$ is an insignificantly singular point.

3.4 Plane curves - metric characteristics

We classify points of a curve $P(t)$ as follows. A point $P(t_0)$ is *a space-like point* or *a time-like point* or *a light-like point* according to the type of the tangent vector $P^{(1)}(t_0)$. A curve is space-like or time-like or light-like if every point of it is of that type. All points of a light-like curve lies on a light-like line. These curves are uninteresting. Space-like and time-like curves are regular. Singular points of a curve are light-like.

A unit tangent vector $\mathbf{t}$ is a vector lying on a tangent line of a curve $P(t)$ which arises by normalization of the tangent vector $P^{(1)}(t)$, i.e. $\mathbf{t}(t) = \frac{P^{(1)}(t)}{|P^{(1)}(t)|}$. *A unit normal vector* $\mathbf{n}$ is such a vector that vectors $\mathbf{t}(t)$ and $\mathbf{n}(t)$ form a positively oriented orthonormal basis, thus $\mathbf{n}(t) = \mathbf{t}^{\perp}(t)$. Orthonormal vectors $\mathbf{t}(t)$ and $\mathbf{n}(t)$ form *the Frenet frame* at every (non-light-like) point of a curve.

A curve $P(t)$ is expressed in *a unit speed parametrization* if $|P^{(1)}(t)| = 1$ for every parameter t. Every curve $P(t)$, $t \in I$ without light-like points can be expressed in a unit speed parametrization $Q(s)$ via a suitable change of parameter.

The curvature $k(t)$ of a curve $P(t)$ at every non-light-like point is $k(t) = \frac{\det\left(P^{(1)}(t), P^{(2)}(t)\right)}{|P^{(1)}(t)|^3}$. In the case of a unit speed parametrization of a curve it is $k(s) = \det(P^{(1)}(s), P^{(2)}(s))$. For such curves hold *the Frenet formulas* $\mathbf{t}^{(1)}(s) = k(s)\mathbf{n}(s)$, $\mathbf{n}^{(1)}(s) = \mathcal{E}\ k(s)\mathbf{t}(s)$.

A regular point $P(t_0)$ is inflexion point of a curve $P(t)$ if $k(t_0) = 0$. A regular point $P(t_0)$ is *an inflexion point of order exactly* n, $n \geq 1$ if $k^{(0)}(t_0) = 0, \dots, k^{(n-1)}(t_0) = 0$ and $k^{(n)}(t_0) \neq 0$. Points which are not inflexion points are called *non-inflexion points*.

A vertex of a curve $P(t)$ is a non-inflexion point $P(t_0)$ such that $k(t_0) \neq 0$, $k^{(1)}(t_0) = 0$. A vertex of a curve is *a vertex of order* n if $k(t_0) \neq 0$, $k^{(1)}(t_0) = 0, \dots, k^{(n)}(t_0) = 0$. A vertex of a curve is *a vertex of order exactly* n if $k(t_0) \neq 0$, $k^{(1)}(t_0) = 0, \dots, k^{(n)}(t_0) = 0$ and $k^{(n+1)}(t_0) \neq 0$. Points which are not vertices are ordinary points of curves.

3.5 Osculating circle of a curve

A circle is a curve expressed by an equation $q(X-S)-\delta r^2 = 0$ where S is centre, $r > 0$ is radius and $\delta = 1$ in the Euclidean plane or $\delta \in \{-1, 1\}$ in the Minkowski plane. The role of circles play (Euclidean) equilateral hyperbolas with light-like asymptotes in the Minkowski plane. For $\delta = 1$ we have *a time-like circle* in the Minkowski plane because every point of this circle is time-like. For $\delta = -1$ we have *a space-like circle* in the Minkowski plane because every point of this circle is space-like. In the Euclidean plane we have only space-like circles and they are the usual Euclidean circles.

At a non-inflexion and not light-like point of a curve there exists a unique circle with centre $S = P(t) - \mathcal{E}\frac{1}{k(t)}\mathbf{n}(t)$ and radius $r = \frac{1}{|k(t)|}$ having contact of order two with the curve at common point. This circle is called *the osculating circle*. It is the tightest attached circle to the curve of all circles having a common point with this curve.

3.6 Classes of functions $P_{n,m}(f)$

To obtain appropriate expressions for derivatives of some concepts of arbitrary order n we introduce some classes of functions.

Let $n \geq 0$ and let P_n be the real vector space of all polynomial functions $\varphi(x_0, \ldots, x_n)$ of $n+1$ real variables. Let $0 \leq m \leq n$ and let $P_{n,m}$ denotes the vector-subspace of P_n generated by monomials $x_0^{k_0} \ldots x_n^{k_n}$, where $k_i \geq 1$ for some $i \geq m$. Thus, a polynomial function $\varphi(x_0, \ldots, x_n) \in P_{n,m}$ if and only if $\varphi(x_0, \ldots, x_n) = 0$ for any vector $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ fulfilling $x_m = \ldots = x_n = 0$. Obviously, this vector subspace can be expressed as $P_{n,m} = \{\varphi(x_0, \ldots, x_n) = \sum_{i=m}^{n} x_i \varphi_i(x_0, \ldots, x_n), \varphi_i(x_0, \ldots, x_n) \in P_n\}$.

Let $f(t)$ be a smooth real function defined on an interval $I \subseteq \mathbb{R}$ and $0 \leq m \leq n$ be integers. Let $j^n f : I \to \mathbb{R}^{n+1}$ be a function assigning the $(n+1)$-tuple of derivatives $(f^{(0)}(t), \ldots, f^{(n)}(t))$ to every $t \in I$. Then the set of all compositions of this function with polynomial functions $\varphi \in P_n$ or $\varphi \in P_{n,m}$ we denote as $P_n(f) = \{\varphi \circ j^n f, \varphi \in P_n\}$ and $P_{n,m}(f) = \{\varphi \circ j^n f, \varphi \in P_{n,m}\}$. Classes $P_{n,m}(f)$ can be written as $P_{n,m}(f) = \Big\{\varphi(f^{(0)}(t), \ldots, f^{(n)}(t)) = \sum_{i=m}^{n} f^{(i)}(t)\varphi_i(f^{(0)}(t), \ldots, f^{(n)}(t)), \varphi_i \in P_n(f)\Big\}$.

For every $n \geq l \geq m \geq 0$ it holds that if $g \in P_{n,m}(f), h \in P_l(f)$ then $gh \in P_{n,m}(f)$ and if $g \in P_{n,m}(f), h \in P_{l,m}(f)$ then $g + h, gh \in P_{\max(n,l),m}(f)$. For every $n \geq m \geq 0$ it holds $d(P_n(f)) \subseteq P_{n+1,1}(f)$ and $d(P_{n,m}(f)) \subseteq P_{n+1,m}(f)$, where $d : P_n(f) \to P_n(f), g \to g^{(1)}$ is a derivative operator.

Studying singular points of evolutes and pedal curves, we intensively use functions from $P_{n,m}(f)$ in cases when $m = 0$ or $m = 1$ and f is the curvature of a curve.

3.7 Summary

We discuss basic notions and properties of plane curves in the Euclidean and in the Minkowski plane concurrently. We emphasize the difference between the two planes consisting in a sign change in the scalar product formula. We introduce a concept of a curve and deal with some affine properties and metric characteristics of curves. Special classes of functions $P_{n,m}(f)$ and their properties are introduced.

4 Evolutes

Evolutes are curves associated to a base curve. They are defined as the locus of centres of all osculating circles of that curve. We express evolutes parametrically and study situation in which singular points occur on them. Classification of finite order singular points of evolutes is discussed in all details. We discuss some properties of evolutes in the Minkowski plane and extend their definition also for light-like points.

The concept of evolute is well-known in geometry of the Euclidean plane. Some properties of evolutes in the Minkowski plane are discussed in [1]. In [11] authors presented interesting results concerning collocation of points of evolutes with respect to the base curve in the Minkowski plane via singularity theory. In paper [3] we re-proved some of their assertions using only elementary tools of differential geometry.

4.1 Definition of evolute

The evolute $E(t)$ of a curve $P(t)$, with its inflexion and light-like points removed, is defined as the locus of centres of all osculating circles of the base curve $P(t)$. Evolute $E(t)$ of a base curve $P(t)$ can be expressed as $E(t) = P(t) - \mathcal{E}\frac{1}{k(t)}\mathbf{n}(t)$.

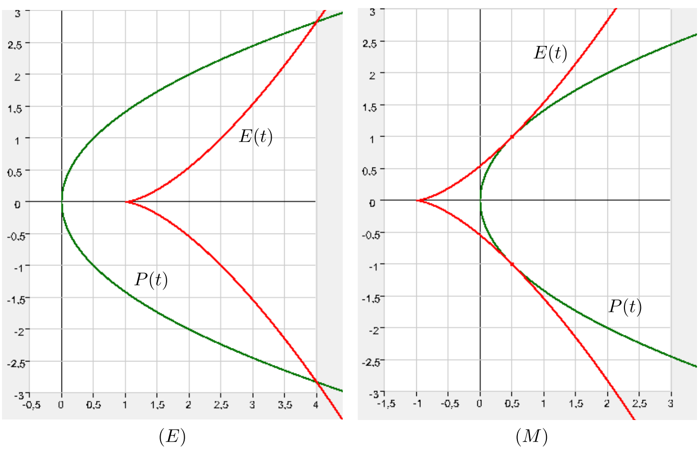

Figure 2: Evolutes $E(t)$ of the parabola $P(t)$ (E) in the Euclidean plane and (M) in the Minkowski plane.

4.2 Singular points of evolute and their classification

Let $P(t)$ be a curve without inflexion and light-like points and $E(t)$ its evolute. The point $E(t_0)$ of evolute is singular if and only if the corresponding point $P(t_0)$ is a vertex of the base curve. If the point $P(t_0)$ is a vertex of order exactly n of the base curve then the corresponding point $E(t_0)$ of the evolute is a singular point of the same order. Moreover, the singular point $E(t_0)$ is an insignificantly singular point for n even and it is a cusp of the first kind for n odd. The tangent line of evolute at the singular point $E(t_0)$ is the normal line of the base curve at $P(t_0)$.

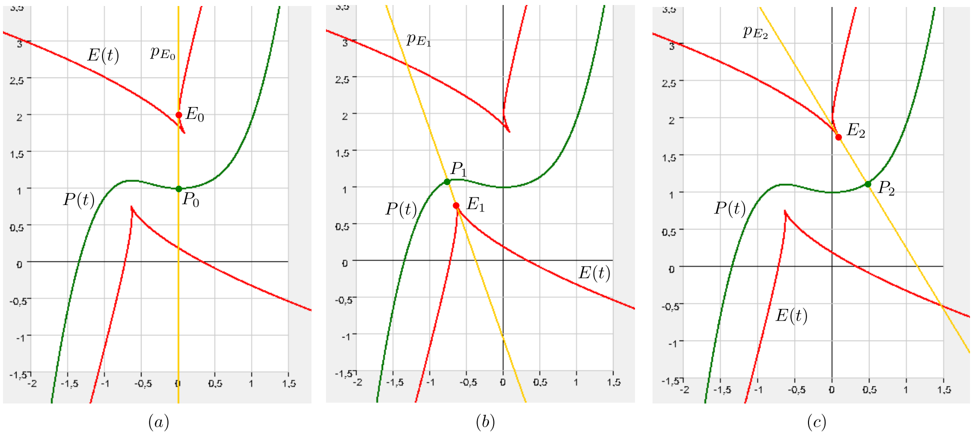

Figure 3: Evolute of the curve $P(t) = (\sinh t, t^5 + \cosh t)$ in the Euclidean plane. (a) E_0 is an insignificantly singular point, (b) E_1 is a cusp of the first kind and (c) E_2 is a cusp of the first kind.

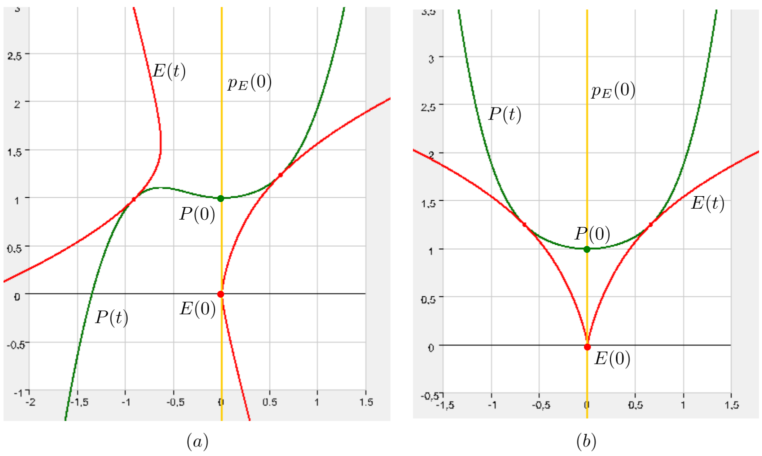

Figure 4: Evolute (a) of the curve $P(t) = (\sinh t, t^5 + \cosh t)$ and (b) of the curve $P(t) = (\sinh t, t^6 + \cosh t)$ in the Minkowski plane. (a) $E(0)$ is an insignificantly singular point and (b) $E(0)$ is a cusp of the first kind.

4.3 Augmented evolutes in the Minkowski plane

Evolute of a base curve $P(t)$, with its inflexion and light-like points removed, is $E(t) = P(t) - \frac{1}{k(t)}\mathbf{n}(t)$. Using definitions of $\mathbf{n}(t)$, $k(t)$ and the symmetry operator in the Minkowski plane we get the formula $E(t) = P(t) - \frac{\langle P^{(1)}(t), P^{(1)}(t)\rangle}{\det(P^{(1)}(t), P^{(2)}(t))}\mathcal{S}P^{(1)}(t)$.

We extend the definition of evolute also for light-like points of the base curve. We use an equi-affine parametrization for that sake. *An equi-affine parametrization* is a parametrization $P(t)$ such that $\det(P^{(1)}(t), P^{(2)}(t)) = 1$ everywhere. Let $P(t)$ be a curve without inflexion points expressed in an equi-affine parametrization. The curve $E(t) = P(t) - \langle P^{(1)}(t), P^{(1)}(t)\rangle \mathcal{S}P^{(1)}(t)$ is called *the augmented evolute* of the base curve $P(t)$. This curve is differentiable also at light-like points.

In the Minkowski plane, the augmented evolute $E(t)$ of a curve $P(t)$ without inflexion points is a regular curve at light-like points with the value $E(t_0) = P(t_0)$ at such point. The base curve and its augmented evolute have common tangent line at light-like point and they lie locally in opposite half-planes considering the common tangent line at light-like point.

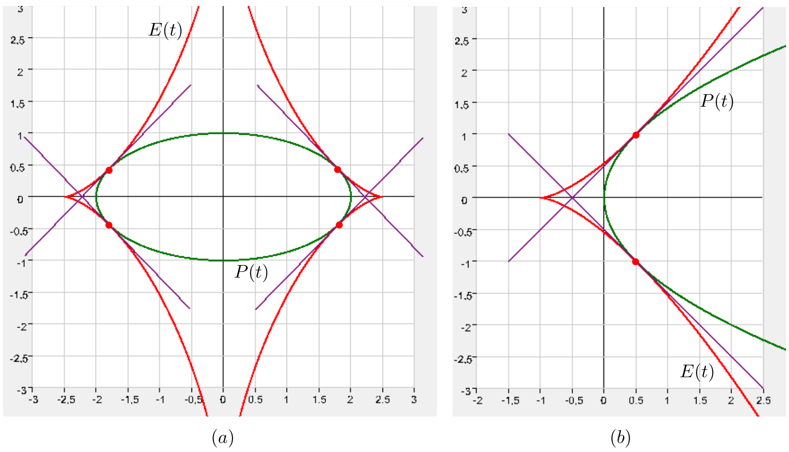

Figure 5: Augmented evolutes $E(t)$ (a) of the ellipse and (b) of the parabola in the Minkowski plane and their common tangent lines with the base curve $P(t)$ at light-like points.

4.4 Summary

Evolutes in the Euclidean and in the Minkowski plane are defined. We determine when singular points on evolute occur and then classify singular points of evolute. We point out connection between singular points of evolute and vertices of the base curve. Other properties of evolute in the Minkowski plane are described via the concept of augmented evolute that extends the definition of evolute also for light-like points.

5 Pedal curves

Pedal curve is defined as a curve associated to a given base curve via orthogonal projections of a given constant point to all tangent lines of the base curve. We describe all situations in which singular points occur on pedal curves. Then classification of finite order singular points of pedal curves is discussed in details.

Singular points of pedal curves in the Euclidean plane are discussed also in [12] and [5]. In both papers authors classified singularities of pedal curves, but only in particular situations on which singular points on pedal curves occur. Singular points of pedal curves in the Minkowski plane are topic of the paper [2]. In the thesis, we provide a full classification of singular points of pedal curves both in the Euclidean and in the Minkowski plane. We do not just convert results from [12] and [5] obtained about singular points of pedal curves in the Euclidean plane as we cover all cases in which singular points on pedal curves occur and classify them in both planes.

5.1 Definition of pedal curve

Let $P(t)$ be a curve without light-like points in the Euclidean plane $\mathbb{E}^2$ or in the Minkowski plane $\mathbb{E}^{1,1}$ and let Q be a fixed point of the plane. At every parameter t construct orthogonal projection $F(t)$ of the point Q into the tangent line $p_P(t)$ of the curve $P(t)$. We call the curve $F(t)$ *pedal curve* of the base curve $P(t)$ with respect to the point Q. We call the point Q *pedal point*. The vector $Q - P(t)$ is called *pedal vector*. The pedal curve $F(t)$ of a base curve $P(t)$ can be expressed as $F(t) = P(t) + \frac{\langle (Q-P(t)), P^{(1)}(t) \rangle}{\langle P^{(1)}(t), P^{(1)}(t) \rangle} P^{(1)}(t)$.

5.2 Singular points of pedal curves and their classification

Let $P(t)$ be a curve without light-like points and let $F(t)$ be its pedal curve with respect to pedal point Q. Then a point $F(t_0)$ of pedal curve is singular if and only if either the corresponding point $P(t_0)$ is an inflexion point of the base curve or the point $P(t_0)$ coincides with the pedal point Q. Thus, a point $F(t_0)$ is singular if and

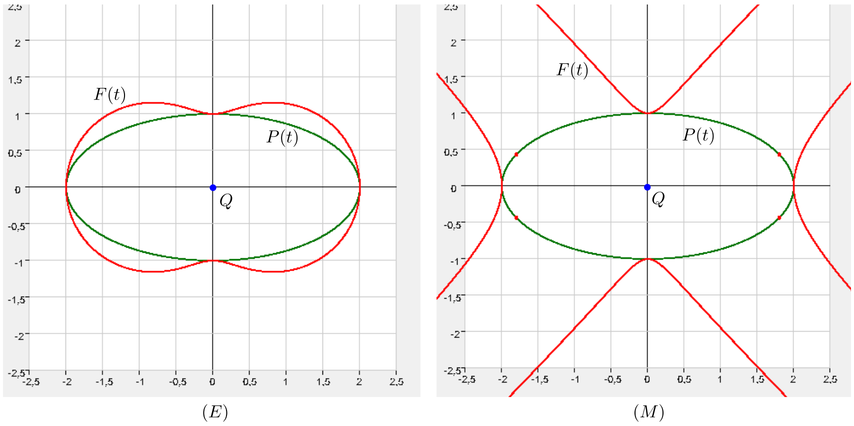

Figure 6: Pedal curves $F(t)$ of the ellipse $P(t)$ with respect to pedal point Q (E) in the Euclidean plane and (M) in the Minkowski plane.

only if exactly one of the following three conditions are fulfilled:
(1) a non-inflexion point $P(t_0)$ of the base curve coincides with the pedal point Q
(2) an inflexion point $P(t_0)$ of the base curve coincides with the pedal point Q
(3) a point $P(t_0)$ of the base curve is an inflexion point and it does not coincide with the pedal point Q

Singular points of pedal curves are classified as follows:
(1) If a non-inflexion point $P(t_0)$ of the base curve coincides with pedal point Q then the point $F(t_0)$ of the pedal curve is a cusp of the first kind. The tangent line of the pedal curve at the singular point $F(t_0)$ is the normal line of the base curve at the point $P(t_0)$.
(2) If $P(t_0)$ is an inflexion point of order exactly n and let $P(t_0) = Q$. Then $F(t_0)$ is an insignificantly singular point if n is odd and $F(t_0)$ is a cusp of the first kind if n is even. The tangent line of pedal curve at the singular point $F(t_0)$ is the normal line of the base curve at $P(t_0)$.
(3) If $P(t_0)$ is an inflexion point of order exactly n and let $P(t_0) \neq Q$. Then $F(t_0)$ is an insignificantly singular point if n is even. If n is odd then $F(t_0)$ is a cusp of the first kind if pedal point does not lie on the tangent line $p_P(t_0)$ of the base curve at $P(t_0)$ and it is a cusp of the second kind if pedal point lies on the tangent line $p_P(t_0)$. In the case when pedal point Q lies on the tangent line of the base curve at $P(t_0)$, the tangent line of pedal curve at the singular point $F(t_0)$ is the normal line of the base curve at $P(t_0)$. In the last case when pedal point Q does not lie on the tangent line of the base curve at $P(t_0)$, the tangent line of pedal curve at the singular point $F(t_0)$ is parallel to the vector $(\langle Q - P(t_0), \mathbf{n}(t_0)\rangle \mathbf{t}(t_0) + \langle Q - P(t_0), \mathbf{t}(t_0)\rangle \mathbf{n}(t_0))$.

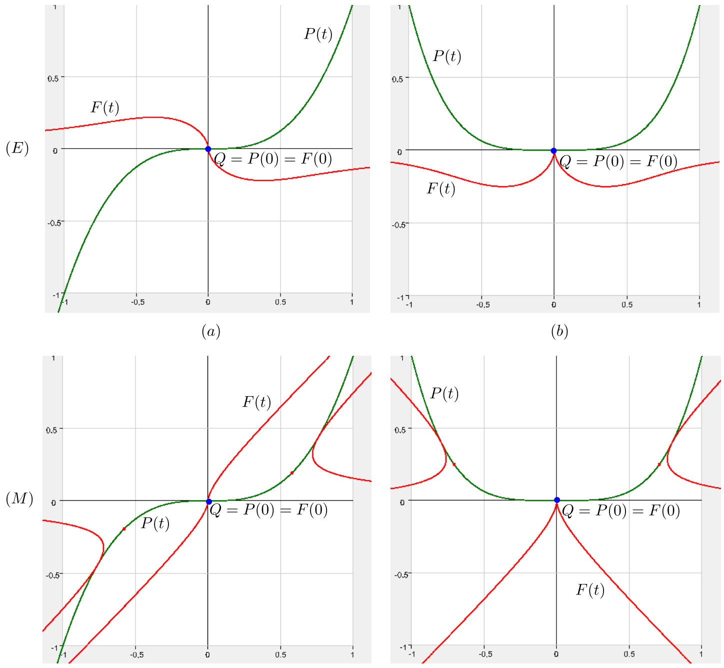

Figure 7: Pedal curve of (a) the curve $P(t) = (t, t^3)$ and (b) the curve $P(t) = (t, t^4)$ (E) in the Euclidean and (M) in the Minkowski plane. (a) $F(0)$ is an insignificantly singular point and (b) $F(0)$ is a cusp of the first kind of pedal curve.

5.3 Summary

We define the notion of pedal curve in the Euclidean and in the Minkowski plane. We localize singular points of pedal curves and distinguish three mutually disjoint situations in which singular points occur on pedal curves. We separately discuss all three cases and fully classify singular points of pedal curves.

6 Conclusion

We presented an overview on geometry of the Euclidean and the Minkowski plane. The affine geometry in these planes is the same. The basic difference consists in scalar product and its change in sign causes modifications in concepts related with it. We discussed the geometry of the two planes concurrently. We presented required affine and metric characteristics of plane curves, again simultaneously in both planes.

We discussed osculating circles of curves in both planes. Contact of a curve and its osculating circle at vertices of higher order were discussed in all detail for the Euclidean plane in the Diploma thesis. Since contact of curves is an affine characteristic

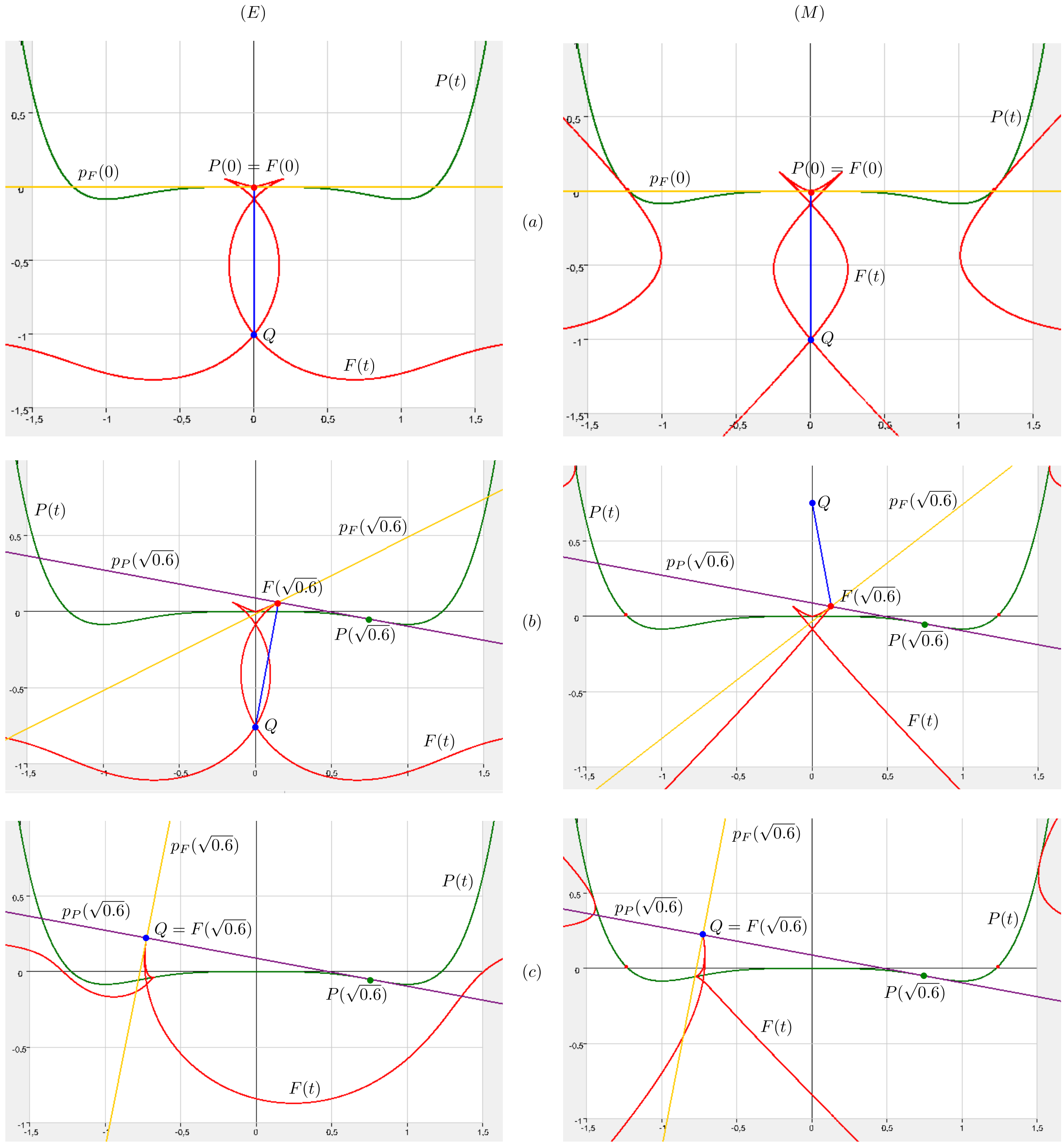

Figure 8: Pedal curve of the curve $P(t) = (t, \frac{1}{6}t^6 - \frac{1}{4}t^4)$ (E) in the Euclidean and (M) in the Minkowski plane. (a) $F(0)$ is an insignificantly singular point, (b) $F(\sqrt{0.6})$ is a cusp of the first kind, (c) $F(\sqrt{0.6})$ is a cusp of the second kind of pedal curve.

of curves which does not change when considering the Minkowski plane instead of the Euclidean one, results achieved for the Euclidean plane holds also for the Minkowski plane. For this reason we considered worthless to re-prove the mentioned results for the Minkowski plane and focused our attention on more interesting areas of research.

We introduced classes of special functions $P_{n,m}(f)$. These classes of functions provide appropriate expressions for derivatives of arbitrary order of some concepts. We discussed the most important properties of the classes of functions $P_{n,m}(f)$.

Singular points of evolutes and their full classification were discussed. Evolute is a curve associated to a given base curve and it is defined as locus of centres of all osculating circles of the base curve. The parametric expression of the evolute was derived from the formula for the centre of osculating circle of the base curve. We studied singular points of evolutes in the Euclidean and in the Minkowski plane in a parallel way. First we detected all such points on evolutes. Then we classified singular points of finite order of evolutes.

We obtained some results about evolutes in the Minkowski plane which have no analogy in the Euclidean plane. We introduced the concept of augmented evolute which extends the definition of evolute also for light-like points and showed some properties of such evolute in the Minkowski plane.

Pedal curves and their singularities in both types of planes were topics of the next research. We provided classification of singular points of pedal curves in all details. Pedal curve is a curve associated to a given base curve. It is constructed by orthogonal projections of a fixed point Q called pedal point to tangent lines of the base curve. We expressed pedal curves via parametrization. We found out that singular points occur on pedal curves in three disjoint situations. We discussed all these situations separately and provided a full classification of singular points of pedal curves.

For the purpose of the dissertation thesis, two original visualization tools for evolutes and pedal curves in the Euclidean and in the Minkowski plane were created. The aim of these visualization tools was to reflect main results of the thesis. All figures occurring in presented work were created by the mentioned tools. The tools animate also processes of creation of evolutes and pedal curves.

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